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Statistic and probability

الاحصاء والاحتمالات

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CHAPTER ONE

Statistic

1-Basic Concepts

- Statistics :is a group of methods used to collect, analyze, present, and interpret data and to make decisions.
- Statistics can be defined as a part of applied mathematics that is concerned with the collection, classification, interpretation, analysis the numerical and categorical data and facts, and drawing conclusions, so as to present the same in a systematic manner.
- is the science of collecting studies to collect, organize, summarize, analyze, and draw conclusions from data

1.1 Introduction

Every day we make decisions that may be personal, business related, or of some other kind. Usually these decisions are made under conditions of uncertainty. Many times, the situations or problems we face in the real world have no precise or definite solution. Statistical methods help us make scientific and intelligent decisions in such situations. Decisions made by using statistical methods are called *educated guesses*. Decisions made without using statistical (or scientific) methods are *pure guesses* and, hence, may prove to be unreliable. For example, opening a large store in an area with or without assessing the need for it may affect its success.

Like almost all fields of study, statistics has two aspects: theoretical and ***applied Theoretical*** or ***mathematical statistics*** deals with the development, derivation, and proof of statistical theorems, formulas, rules, and laws. *Applied statistics* involves the applications of those theorems, formulas, rules, and laws to solve real-world problems. This text is concerned with applied statistics and not with theoretical statistics. By the time you finish studying this course , you will have learned how to think statistically and how to make educated guesses.

1.2 Population

A population consists of all elements ,individuals, items, or objects whose characteristics are being studied. The population that is being studied is also called the target population.

1.3 Sample

- is a group of subjects selected from a population.
- A portion of the population selected for study is referred to as a *sample*

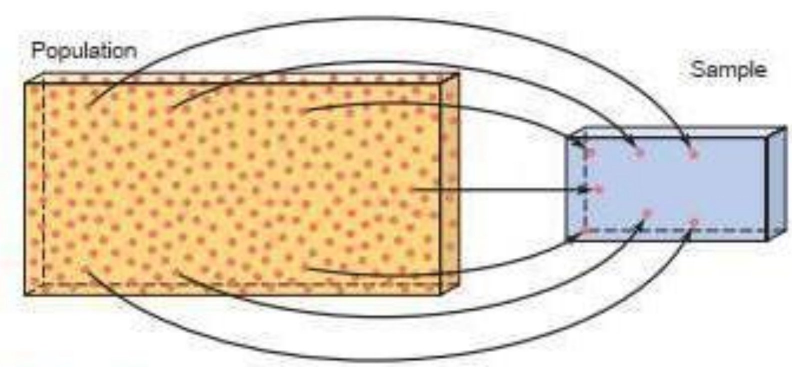


Figure 1.1 Population and sample.

1.3.1 Type of samples

A- *Simple Random Sampling*

In simple random sampling technique, every item in the population has an equal and likely chance of being selected in the sample. Since the item selection entirely depends on the **chance**, this method is known as "**Method of chance Selection**". As the sample size is large, and the item is chosen randomly, it is known as "**Representative Sampling**".

***Example*:**

Suppose we want to select a simple random sample of 200 students from a school. Here, we can assign a number to every student in the school database from 1 to 500 and use a random number generator to select a sample of 200 numbers.

منهجي B-Systematic Sampling

In the systematic sampling method, the items are selected from the target population by selecting the random selection point and selecting the other methods after a fixed sample interval. It is calculated by dividing the total population size by the desired population size.

***Example*:**

Suppose the names of 300 students of a school are sorted in the reverse alphabetical order. To select a sample in a systematic sampling method, we have to choose some 15 students by randomly selecting a starting number, say 5. From number 5 onwards, will select every 15th person from the sorted list. Finally, we can end up with a sample of some students.

الطبقي C-Stratified Sampling

In a stratified sampling method, the total population is divided into smaller groups to complete the sampling process. The small group is formed based on a few characteristics in the population. After separating the population into a smaller group, the statisticians randomly select the sample.

Example

For example, there are three bags (A, B and C), each with different balls. Bag A has 50 balls, bag B has 100 balls, and bag C has 200 balls. We have to choose a sample of balls from each bag proportionally. Suppose 5 balls from bag A, 10 balls from bag B and 20 balls from bag C.

D-Clustered Sampling

In the clustered sampling method, the cluster or group of people are formed from the population set. The group has similar significatory characteristics. Also, they have an equal chance of being a part of the sample. This method uses simple random sampling for the cluster of population.

***Example*:**

An educational institution has ten branches across the country with almost the number of students. If we want to collect some data regarding facilities and other things, we can’t travel to every unit to collect the required data. Hence, we can use random sampling to select three or four branches as clusters.

All these four methods can be understood in a better manner with the help of the figure given below. The figure contains various examples of how samples will be taken from the population using different techniques.

1.4 Variable

: A *variable* is a characteristic under study that assumes different values for different elements. In contrast to a variable, the value of a *constant* is fixed.

Type of Variable

A-Quantitative Variables

Some variables (such as the price of a home) can be measured numerically, whereas others (such as hair color) cannot. The first is an example of a ***quantitative variable*** and the second that of a **qualitative variable.**

Quantitative Variable A variable that can be measured numerically is called a quantitative variable. The data collected on a quantitative variable are called quantitative data ,Incomes, heights, gross sales, prices of homes, number of cars owned, and number of accidents are examples of quantitative variables because each of them can be **expressed numerically**. For instance, the income of a family may be $81,520.75 per year, the gross sales for a company may be $567 million for the past year, and so forth. Such quantitative variables may be classified as either discrete variables *or* continuous variables.

Discrete Variables

The values that a certain quantitative variable can assume may be countable or non- countable. For example, we can count the number of cars owned by a family, but we cannot count the height of a family member. A variable that assumes countable values is called a discrete variable. Note that there are no possible intermediate values between consecutive values of a discrete variable.

Discrete Variable: A variable whose values are countable is called a ***discrete variable***. In other words, a discrete variable can assume only certain values with no intermediate values.

For example, the number of cars sold on any day at a car dealership is a discrete variable because the number of cars sold must be 0, 1, 2, 3, . . . and we can count it. The number of cars sold cannot be between 0 and 1, or between 1 and 2. Other examples of discrete variables are the number of people visiting a bank on any day, the number of cars in a parking lot, the number of cattle owned by a farmer, and the number of students in a class.

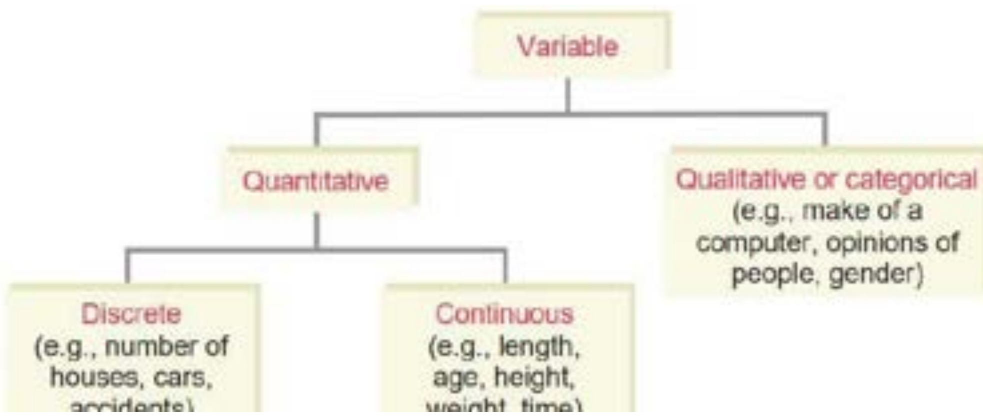
Continuous Variables

Some variables cannot be counted, and they can assume any numerical value between **two numbers**. Such variables are called ***continuous variables.***

Continuous Variable :A variable that can assume any numerical value over a **certain interval** or intervals is called a continuous variable.

The time taken to complete an examination is an example of a continuous variable because it can assume any value, let us say, between 30 and 60 minutes. The time taken may be 42.6 minutes, 42.67 minutes, or 42.674 minutes. (Theoretically, we can measure time as precisely as we want.) Similarly, the height of a person can be measured to the tenth of an inch or to the hundredth of an inch. However, neither time nor height can be counted in a discrete fashion. Other examples of continuous variables are weights of people, amount of soda in a 12-ounce can (note that a can does not contain exactly 12 ounces of soda), and yield of potatoes (in pounds) per acre. Note that any variable that involves money is considered a continuous variable

Qualitative or Categorical Variable A variable that cannot assume a numerical value but can be classified into two or more nonnumeric categories is called a **qualitative *or* categorical variable**. The data collected on such a variable are called qualitative data. For example, the status of an undergraduate college student is a qualitative variable because a student can fall into any one of four categories: freshman, sophomore, junior, or senior. Other examples of qualitative variables are the gender of a person, the brand of a computer, the opinions of people, and the make of a car.



1.5 Raw data

When data are collected, the information obtained from each member of a population or sample is recorded in the sequence in which it becomes available. This sequence of data recording is random and unranked. Such data, before they are grouped or ranked, are called *raw data*.

Suppose we collect information on the ages (in years) of 50 students selected from a university. The data values, in the order they are collected, are recorded in below Table. For instance, the first student's age is 21, the second student's age is 19 (second number in the first row), and so forth. The data in Table 2.1 are quantitative raw data.

Table 2.1 Ages of 50 Students

21	19	24	25	29	34	26	27	37	33
18	20	19	22	19	19	25	22	25	23
25	19	31	19	23	18	23	19	23	26
22	28	21	20	22	22	21	20	19	21
25	23	18	37	27	23	21	25	21	24

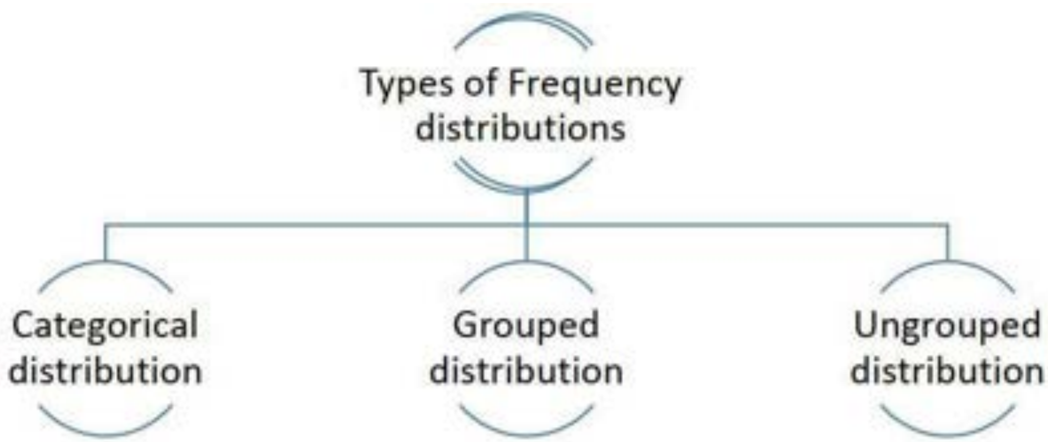
Suppose we ask the same 50 students about their student status. The responses of the students are recorded in below Table In this table, F, SO, J, and SE are the abbreviations for freshman, sophomore, junior, and senior, respectively. This is an example of qualitative (or categorical) raw data.

Table 2.2 Status of 50 Students

J	F	SO	SE	J	J	SE	J	J	J
F	F	J	F	F	F	SE	SO	SE	J
J	F	SE	SO	SO	F	J	F	SE	SE
SO	SE	J	SO	SO	J	J	SO	F	SO
SE	SE	F	SE	J	SO	F	J	SO	SO

1.6 Data Organization

frequency distribution



A frequency distribution : is the organization of raw data in table form, using classes and frequencies. When data are collected in original form, they are called raw data.

2	5	8	7	2	2
6	8	5	2	5	7
4	5	6	2	8	6

Score	f
8	3
7	2
6	3
5	4
4	1
2	5

Categorical Frequency Distribution

Categorical Frequency Distribution: can be used for data that can be placed in specific categories, such as nominal- or ordinal-level data.

Example

Raw Data: A,B,B,AB,O O,O,B,AB,B B,B,O,A,O A,O,O,O,AB AB,A,O,B,A

Percent= $\frac{f}{n} * 100$

Class	Tally	Frequency (f)	Percent
A	~~IIII~~	5	20
B	~~IIII~~ II	7	28
O	~~IIII~~ IIII	9	36
AB	IIII	4	16
		n=25	100

Grouped Frequency Distribution

Grouped frequency distributions can be used when the range of values in the data set is very large. The data must be grouped into **classes** that are more than one unit in width. For example, the life of boat batteries in hours.

- The smallest and largest possible data values in a class are **the *lower* and *upper class limits***. ***Class boundaries*** separate the classes.
- To find a class boundary, average the upper class limit of one class and the lower class limit of the next class.

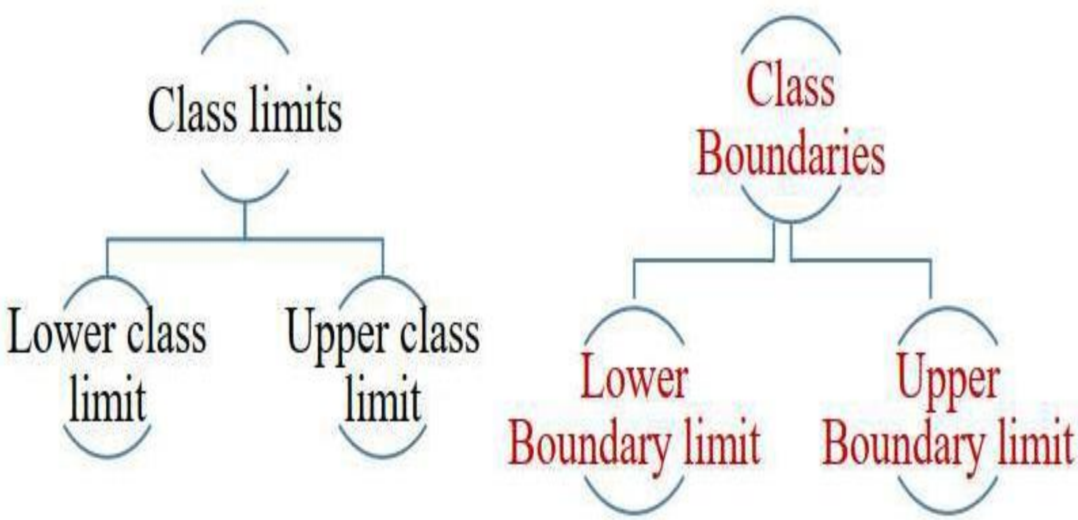
➢ The *class width* can be calculated by subtracting

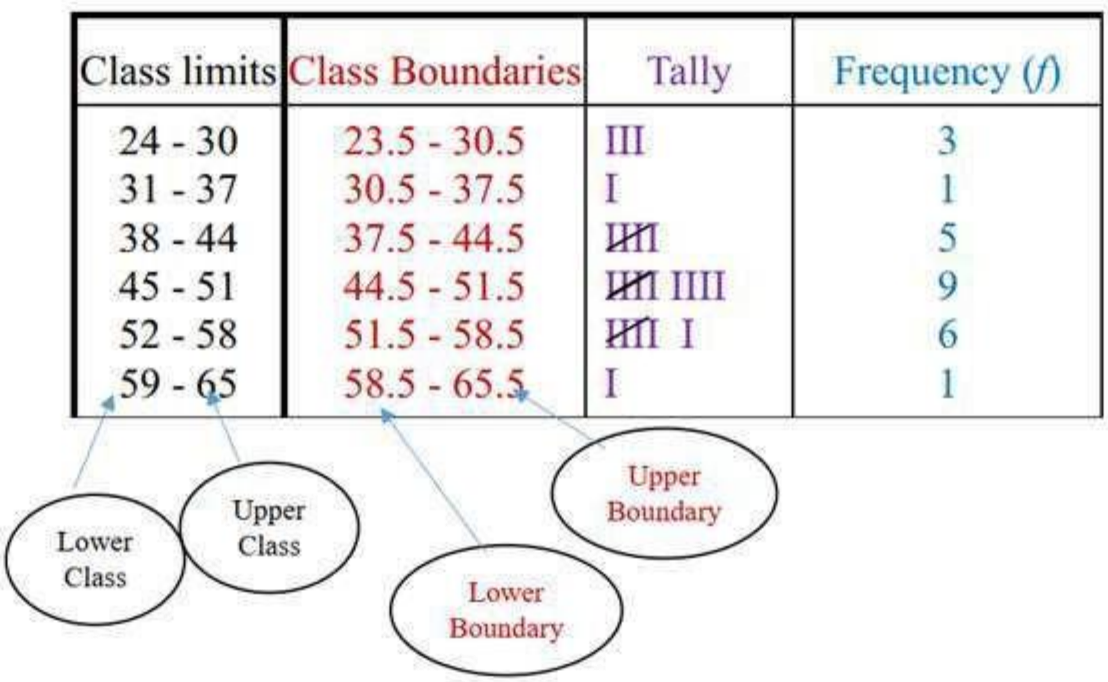
✓ successive lower class limits (or boundaries)
✓ successive upper class limits (or boundaries)
✓ upper and lower class boundaries.

✓ The *class midpoint Xm* can be calculated by averaging upper and lower class limits (or boundaries).

Calculating Class Midpoint or Mark

$$\text{Class midpoint or mark} = \frac{\text{Lower limit} + \text{Upper limit}}{2}$$





Class limits	Class Boundaries	Tally	Frequency (*f*)
24 - 30	23.5 - 30.5	III	3
31 - 37	30.5 - 37.5	I	1
38 - 44	37.5 - 44.5	~~IIII~~	5
45 - 51	44.5 - 51.5	~~IIII~~ IIII	9
52 - 58	51.5 - 58.5	~~IIII~~ I	6
59 - 65	58.5 - 65.5	I	1

-In the life of boat batteries example, the values 24 and 30 of the first class are the *class limits.*

- The *lower class* limit is 24 and the *upper class* limit is 30.

- *The* ***Class boundaries*** are used to separate the classes. So that there are no gaps in the frequency distribution.

-The difference between the two boundaries of a class gives the *class width*. The class width is also called the *class size*.

- **Lower boundary= lower limit - 0.5**
- **Upper boundary= upper limit + 0.5**

- Class limits should have the same decimal place value as the data, but the class boundaries should have one additional place value and end in a 5.

For example: Class limit 7.8 – 8.8

Class boundary 7.75 – 8.85

➢Lower boundary= lower limit - 0.05

=7.8- 0.05 =7.75

➢Upper boundary= upper limit + 0.05

– =8.8+0.05=8.85

Finding Class Width

Class width = Upper boundary − Lower boundary

Class width = lower of second class limit- lower of first class limit

Or

Class width = Upper of second class limit- Upper of first class limit

Calculation of Class Width

$$\text{Approximate class width} = \frac{\text{Largest value} - \text{Smallest value}}{\text{Number of classes}}$$

Class width: 31 – 24 =7

The class **midpoint** (Xm) is found by adding the lower and upper class limit (or boundary) and dividing by 2.

$$Xm = \frac{lwer\ limit + uper\ limit}{2} \quad \text{or} \quad Xm = \frac{lwer\ \text{boundary} + uper\ \text{boundary}}{2}$$

For example.

$$Xm=\frac{24+30}{2}=27 \quad , \quad Xm=\frac{23.5+30.5}{2}=27$$

- Find the boundaries for the following class limits:
  - 44 - 37
  - 10.3 - 11.5
  - 22.2 – 23.0
  - 547.04 - 553.20
- Find the class width for the following class limits:
  - 37 – 44
  - 45 – 52
  - 625 – 654
  - 655 - 684
- Find the class width for the following class boundaries:
  - 10.5 – 11.5
  - 22.15 – 27.15

Rules for Classes in Grouped Frequency Distributions

1. There should be 5-20 classes.
2. The class width should be an odd number.
3. The classes must be mutually exclusive.

Age
10 – 20
20 – 30
30 – 40
40 – 50

Better wat to construct a frequency distribution

Age
10 – 20
21 – 31
32 – 42
43 – 53

4. The classes must be continuous.
5. The classes must be exhaustive.
6. The classes must be equal in width (except in open-ended distributions).

Procedure for Constructing a Grouped Frequency Distribution

- STEP 1 Determine the classes.

  - Find the highest and lowest value
  - Find the range
  - Select the number of classes desired.
  - Find the width by divided the range by the number of
  - classes and rounding up.
  - Select a starting point (usually the lowest value), add the
  - width to get the lower limits.
  - Find the upper class limits.
  - Find the boundaries.

- STEP 2 Tally the data.
- STEP 3 Find the frequencies.
- STEP 4 Find the cumulative frequencies by keeping a running

total of the frequencies.

Example

The following data represent the record high temperatures for each of the 50 states. Construct a grouped frequency distribution for the data **using 7 classes**.

112	100	127	120	134	118	105	110	109	112
110	118	117	116	118	122	114	114	105	109
107	112	114	115	118	117	118	122	106	110
116	108	110	121	113	120	119	111	104	111
120	113	120	117	105	110	118	112	114	114

STEP 1 Determine the classes. Find the class width by dividing the range by the number of classes 7.

Range = High – Low

= 134 – 100 = 34

width= $\frac{Range}{7}$ **=34/7=5.**

Note: Rounding Rule: Always round up if a remainder.

STEP 2 Tally the data.

STEP 3 Find the frequencies

Class Limits	Class Boundaries	Frequency	Cumulative Frequency
100 - 104	99.5 - 104.5	2	
105 - 109	104.5 - 109.5	8	
110 - 114	109.5 - 114.5	18	
115 - 119	114.5 - 119.5	13	
120 - 124	119.5 - 124.5	7	
125 - 129	124.5 - 129.5	1	
130 - 134	129.5 - 134.5	1	

STEP 4 Find the cumulative frequencies by keeping a running total of the frequencies.

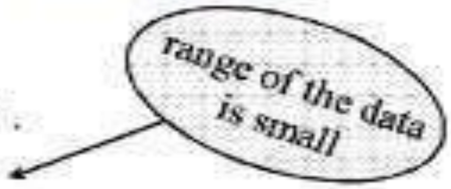
Class Limits	Class Boundaries	Frequency	Cumulative Frequency
100 - 104	99.5 - 104.5	2	2
105 - 109	104.5 - 109.5	8	10
110 - 114	109.5 - 114.5	18	28
115 - 119	114.5 - 119.5	13	41
120 - 124	119.5 - 124.5	7	48
125 - 129	124.5 - 129.5	1	49
130 - 134	129.5 - 134.5	1	50

Ungrouped Frequency Distribution.

Example : The data shown here represent the number of miles per gallon that 30 selected four-wheel- drive sports utility vehicles obtained in city driving.

12	17	12	14	16	18
16	18	12	16	17	15
15	16	12	15	16	16
12	14	15	12	15	15
19	13	16	18	16	14

STEP 1 Determine the classes.
The rang of the data set is small .



Range = High – Low

$= 19 - 12 = 7$

So the class consisting of the single data value can be used. They are 12,13,14,15,16,17,18,19.

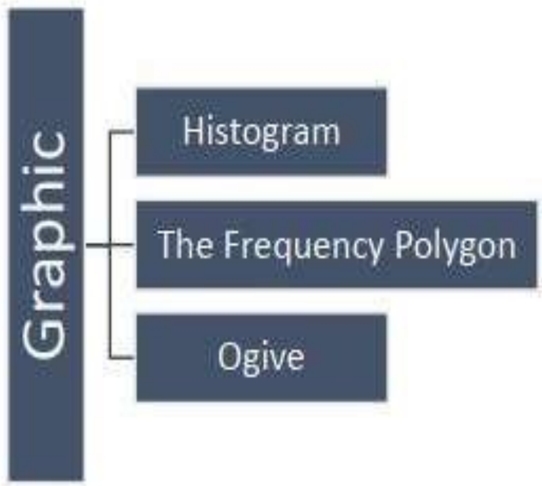
- This type of distribution is called ungrouped frequency distribution

STEP 2 Tally the data.
STEP 3 Find the frequencies.

Class Limits	Class Boundaries	Frequency	Cumulative Frequency
12	11.5-12.5	6	0
13	12.5-13.5	1	6
14	13.5-14.5	3	7
15	14.5-15.5	6	10
16	15.5-16.5	8	16
17	16.5-17.5	2	24
18	17.5-18.5	3	26
19	18.5-19.5	1	29
			30

Graphing Grouped Data

Grouped (quantitative) data can be displayed in.



Purpose of graphs in statistics is to convey the **data** to the viewers **in pictorial form**

- **Easier** for most people to understand the **meaning of data** in form of graphs
- They can also be used to discover a **trend or pattern** in a situation over a period of time
- Useful in **getting** the audience's **attention** in a publication or a speaking presentation

➢ Histogram

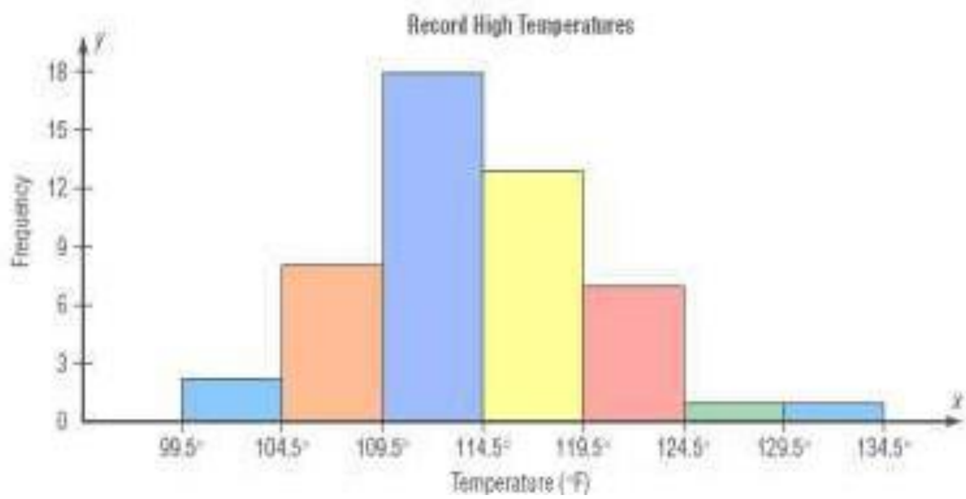
The histogram is a graph that displays the data by using contiguous vertical bars (unless the frequency of a class is 0) of various heights to represent the frequencies of the classes.

❑The <u>class boundaries</u> are represented on the horizontal axis

Example 2-4: Construct a histogram to represent the data for the record high temperatures for each of the 50 states (see Example 2–2 for the data).

Class Limits	Class Boundaries	Frequency
100 - 104	99.5 - 104.5	2
105 - 109	104.5 - 109.5	8
110 - 114	109.5 - 114.5	18
115 - 119	114.5 - 119.5	13
120 - 124	119.5 - 124.5	7
125 - 129	124.5 - 129.5	1
130 - 134	129.5 - 134.5	1

Histograms use class boundaries and frequencies of the classes



Measurement of central tendency

We often represent a data set by numerical summary measures, usually called the *typical values.* A measure of central tendency gives the center of a histogram or a frequency distribution curve. This section discusses three different measures of central tendency: the mean, the median, and the mode;

1- Mean (ungrouped data)

The *mean*, also called the arithmetic mean, is the most frequently used measure of central tendency. In this section we will use the words mean and average synonymously. For ungrouped data, the mean is obtained by dividing the sum of all values by the number of values in the data set:

The mean calculated for sample data is denoted by $\bar{X}$ (read as "*x* bar"), and the mean calculated for population data is denoted by m (Greek letter *mu*). We know from the discussion in that the number of values in a data set is denoted by *n* for a **sample** and by *N* for a **population**. In we learned that a variable is denoted by *x*, and the sum of all values of *x* is denoted by $\sum$ Using these notations, we can write the following formulas for the mean.

Calculating Mean for Ungrouped Data The *mean for ungrouped data* is obtained by dividing the sum of all values by the number of values in the data set. Thus,

$$\text{Mean for population data:} \quad \mu = \frac{\Sigma x}{N}$$

$$\text{Mean for sample data:} \quad \bar{x} = \frac{\Sigma x}{n}$$

where Σx is the sum of all values, N is the population size, n is the sample size, μ is the population mean, and $\bar{x}$ is the sample mean.

Mean

- The mean is the quotient of the sum of the values and the total number of values.
- The symbol $\bar{X}$ is used for sample mean.

$$\bar{X} = \frac{X_1 + X_2 + X_3 + \cdots + X_n}{n} = \frac{\sum X}{n}$$

- For a population, the Greek letter μ (mu) is used for the mean.

$$\mu = \frac{X_1 + X_2 + X_3 + \cdots + X_N}{N} = \frac{\sum X}{N}$$

Days off per Year

The data represent the number of days off per year for a sample of individuals selected from nine different countries. Find the mean.

20, 26, 40, 36, 23, 42, 35, 24, 30

$$\bar{X} = \frac{X_1 + X_2 + X_3 + \cdots + X_n}{n} = \frac{\sum X}{n}$$

$$\bar{X} = \frac{20+26+40+36+23+42+35+24+30}{9} = \frac{276}{9} = 30.7$$

The mean number of days off is 30.7 years.

Example 2

Below table lists the total sales (rounded to billions of dollars) of six U.S. companies for 2008. Find the 2008 mean sales for these six companies

Company	Total Sales (billions of dollars)
General Motors	149
Wal-Mart Stores	406
General Electric	183
Citigroup	107
Exxon Mobil	426
Verizon Communication	97

Solution The variable in this example is the 2008 total sales for a company. Let us denote this variable by x. Then, the six values of x are

$$x_1 = 149, \quad x_2 = 406, \quad x_3 = 183, \quad x_4 = 107, \quad x_5 = 426, \quad \text{and} \quad x_6 = 97$$

where $x_1 = 149$ represents the 2008 total sales of General Motors, $x_2 = 406$ represents the 2008 total sales of Wal-Mart Stores, and so on. The sum of the 2008 sales for these six companies is

$$\Sigma x = x_1 + x_2 + x_3 + x_4 + x_5 + x_6$$

$$= 149 + 406 + 183 + 107 + 426 + 97 = 1368$$

Note that the given data include only six companies. Hence, they represent a sample. Because the given data set contains six companies, $n = 6$. Substituting the values of Σx and n in the sample formula, we obtain the mean 2008 sales of the six companies:

$$\bar{x} = \frac{\Sigma x}{n} = \frac{1368}{6} = \mathbf{228} = \mathbf{\$228\ Billion}$$

Thus, the mean 2008 sales of these six companies was 228, or \$228 billion.

EXAMPLE

The following are the ages (in years) of all eight employees of a small company:

53 32 61 27 39 44 49 57

Find the mean age of these employees.

Solution Because the given data set includes *all* eight employees of the company, it represents the population. Hence, $N = 8$. We have

$$\Sigma x = 53 + 32 + 61 + 27 + 39 + 44 + 49 + 57 = 362$$

The population mean is

$$\mu = \frac{\Sigma x}{N} = \frac{362}{8} = \mathbf{45.25\ years}$$

Thus, the mean age of all eight employees of this company is 45.25 years, or 45 years and 3 months.

2- MEDIAN

- The median is the midpoint of the **data array**. The symbol for the median is **MD**.

Finding the median

Step 1 Arrange the data values in ascending order.

Step 2 determine the number of values in the data set.

Step 3 a. If n is odd, select the middle data value as the median.

b. If n is even, find the mean of the two middle values. That is, add them and divide the sum by 2.

Definition

Median The *median* is the value of the middle term in a data set that has been ranked in increasing order.

As is obvious from the definition of the median, it divides a ranked data set into two equal parts. The calculation of the median consists of the following two steps:

1. Rank the data set in increasing order.
2. Find the middle term. The value of this term is the median.[1]

Note that if the number of observations in a data set is *odd*, then the median is given by the value of the middle term in the ranked data. However, if the number of observations is *even*, then the median is given by the average of the values of the two middle terms.

Example 1 \Police Officers Killed

The number of police officers killed in the line of duty over the last 11 years is shown. Find the median.

177 153 122 141 189 155 162 165 149 157 240

Sol\

Sort in ascending order

122, 141, 149, 153, 155, 157, 162, 165, 177, 189, 240

Select the middle value.

MD = 157

The median is 157 rooms.

***Example 2*/Tornadoes in the U.S.**

The number of tornadoes that have occurred in the United States over an 8-year period follows. Find the median.

684, 764, 656, 702, 856, 1133, 1132, 1303 Find the average of the two middle values.

656, 684, 702, 764, 856, 1132, 1133, 1303

MD= $\frac{764+856}{2} = \frac{1620}{2}$ = 810

The median number of tornadoes is 810

■ EXAMPLE

The following data give the prices (in thousands of dollars) of seven houses selected from all houses sold last month in a city.

312 257 421 289 526 374 497

Find the median.

Solution First, we rank the given data in increasing order as follows:

257 289 312 374 421 497 526

Since there are seven homes in this data set and the middle term is the fourth term, the median is given by the value of the fourth term in the ranked data.

257 289 312 374 421 497 526

↑ Median (374)

Thus, the median price of a house is 374, or **$374,000**.

Table gives the 2008 profits (rounded to billions of dollars) of 12 companies selected from all over the world.

Table 3.3 Profits of 12 Companies for 2008

Company	2008 Profits (billions of dollars)
Merck & Co	8
IBM	12
Unilever	7
Microsoft	17
Petrobras	14
Exxon Mobil	45
Lukoil	10
AT&T	13
Nestlé	17
Vodafone	13
Deutsche Bank	9
China Mobile	11

Find the median for these data.

Solution First we rank the given profits as follows:

7 8 9 10 11 12 13 13 14 17 17 45

There are 12 values in this data set. Because there is an even number of values in the data set, the median is given by the average of the two middle values. The two middle values are the sixth and seventh in the foregoing list of data, and these two values are 12 and 13. The median, which is given by the average of these two values, is calculated as follows.

7 8 9 10 11 **12 13** 13 14 17 17 45

↑ Median

$$\text{Median} = \frac{12 + 13}{2} = \frac{25}{2} = 12.5 = \mathbf{\$12.5\ billion}$$

Thus, the median profit of these 12 companies is $12.5 billion.

The median gives the center of a histogram, with half of the data values to the left of the median and half to the right of the median. The advantage of using the median as a measure of central tendency is that it is not influenced by outliers. Consequently, the median is preferred over the mean as a measure of central tendency for data sets that contain outliers.

3- Mode

- The mode is the value that occurs most often in a data set.
- It is sometimes said to be the most typical case.
- There may be no mode, one mode (unimodal), two modes (bimodal), or many modes (multimodal).

Definition

Mode The *mode* is the value that occurs with the highest frequency in a data set.

EXAMPLE

The following data give the speeds (in miles per hour) of eight cars that were stopped on I-95 for speeding violations.

77 82 74 81 79 84 74 78

Find the mode.

Solution In this data set, 74 occurs twice, and each of the remaining values occurs only once. Because 74 occurs with the highest frequency, it is the mode. Therefore,

Mode = **74 miles per hour**

EXAMPLE

Last year's incomes of five randomly selected families were $76,150, $95,750, $124,985, $87,490, and $53,740. Find the mode.

Solution Because each value in this data set occurs only once, this data set contains **no mode.**

EXAMPLE

Refer to the data on 2008 profits of 12 companies given in Table 3.3 of Example 3–5. Find the mode for these data.

Solution In the data given in Example 3–5, each of the two values 13 and 17 occurs twice, and each of the remaining values occurs only once. Therefore, that data set has two modes: $13 billion and $17 billion.

EXAMPLE

The ages of 10 randomly selected students from a class are 21, 19, 27, 22, 29, 19, 25, 21, 22, and 30 years, respectively. Find the mode.

Solution This data set has three modes: **19**, **21**, and **22**. Each of these three values occurs with a (highest) frequency of 2.

One advantage of the mode is that it can be calculated for both kinds of data—quantitative and qualitative—whereas the mean and median can be calculated for only quantitative data.

Example

Find the mode of the signing bonuses of eight NFL players for a specific year. The bonuses in millions of dollars are

18.0, 14.0, 34.5, 10, 11.3, 10, 12.4, 10

You may find it easier to sort first. 10, 10, 10, 11.3, 12.4, 14.0, 18.0, 34.5 Select the value that occurs the most.

The mode is 10 million dollars.

Example ---

The data show the number of licensed nuclear reactors in the United States for a recent 15-year period. Find the mode.

104 and 109 both occur the most. The data set is said to be bimodal.

The mode is 10 million dollars.

104 104 104 104 104 107 109 109 109 110

109 111 112 111 109

The modes are 104 and 109

Mean for Grouped Data

We learned that the mean is obtained by dividing the sum of all values by the number of values in a data set. However, if the data are given in the form of a **frequency table**, we no longer know the values of individual observations. Consequently, in such cases, we cannot obtain the sum of individual values. We find an approximation for the sum of these values using the procedure explained in the next paragraph and example. The formulas used to calculate the mean for grouped data follow.

Step 1 Make a table as shown

A Class	B Frequency f	C Midpoint X_m	D $f \cdot X_m$

Step 2 Find the midpoints of each class and place them in column C.

Step 3 multiply the frequency by the midpoint for each class, and place the product in column D.

Step 4 Find the sum of column D.

Step 5 Divide the sum obtained in column D by the sum of frequencies obtained in column B.

The formula for the mean is

Calculating Mean for Grouped Data

Mean for population data: $\mu = \frac{\Sigma mf}{N}$

Mean for sample data: $\bar{x} = \frac{\Sigma mf}{n}$

where m is the midpoint and f is the frequency of a class.

To calculate the mean for grouped data, first find the midpoint of each class and then multiply the midpoints by the frequencies of the corresponding classes. The sum of these products, denoted by gives an approximation for the sum of all values. To find the value of the mean, divide this sum by the total number of observations in the data.

example

Using the frequency distribution for Example below, find the mean. The data represent the number of miles run during one week for a sample of 20 runners.

Solution

The procedure for finding the mean for grouped data is given here.

Step 1 Make a table as shown.

A Class	B Frequency f	C Midpoint X_m	D $f \cdot X_m$
5.5–10.5	1		
10.5–15.5	2		
15.5–20.5	3		
20.5–25.5	5		
25.5–30.5	4		
30.5–35.5	3		
35.5–40.5	2		
	$n = 20$		

Step 2 Find the midpoints of each class and enter them in column C

$$X_m = \frac{5.5 + 10.5}{2} = 8 \qquad \frac{10.5 + 15.5}{2} = 13 \qquad \text{etc.}$$

Step 3 For each class, multiply the frequency by the midpoint, as shown, and place the product in column D.
1 * 8 =8 2 * 13 = 26 etc. The completed table is shown here.

A Class	B Frequency f	C Midpoint X_m	D $f \cdot X_m$
5.5–10.5	1	8	8
10.5–15.5	2	13	26
15.5–20.5	3	18	54
20.5–25.5	5	23	115
25.5–30.5	4	28	112
30.5–35.5	3	33	99
35.5–40.5	2	38	76
	$n = 20$		$\Sigma f \cdot X_m = 490$

Step 4 Find the sum of column D.

Step 5 Divide the sum by n to get the mean.

$$\overline{X} = \frac{\Sigma f \cdot X_m}{n} = \frac{490}{20} = 24.5 \text{ miles}$$

Measurements of variation

Variance and Standard Deviation

The **standard deviation** is the most-used measure of dispersion. The value of the standard deviation tells how closely the values of a data set are clustered around the mean. In general, a lower value of the standard deviation for a data set indicates that the values of that data set are spread over a relatively smaller range around the mean. In contrast, a larger value of the standard deviation for a data set indicates that the values of that data set are spread over a relatively larger range around the mean.

The *standard deviation is obtained by taking the positive square root of the* **variance**. The variance calculated for population data is denoted by σ^2 (read as *sigma squared*),[2] and the variance calculated for sample data is denoted by s^2. Consequently, the standard deviation calculated for population data is denoted by σ, and the standard deviation calculated for sample data is denoted by s. Following are what we will call the *basic formulas* that are used to calculate the variance:[3]

$$\sigma^2 = \frac{\Sigma(x-\mu)^2}{N} \quad \text{and} \quad s^2 = \frac{\Sigma(x-\bar{x})^2}{n-1}$$

where σ^2 is the population variance and s^2 is the sample variance.

The quantity $x - \mu$ or $x - \bar{x}$ in the above formulas is called the *deviation* of the x value from the mean. The sum of the deviations of the x values from the mean is always zero; that is, $\Sigma(x-\mu) = 0$ and $\Sigma(x-\bar{x}) = 0$.

For example, suppose the midterm scores of a sample of four students are 82, 95, 67, and 92, respectively. Then, the mean score for these four students is

$$\bar{x} = \frac{82 + 95 + 67 + 92}{4} = 84$$

The deviations of the four scores from the mean are calculated in Table below. As we can observe from the table, the sum of the deviations of the *x* values from the mean is zero; For this reason we square the deviations to calculate the variance and standard deviation.

x	$x - \bar{x}$
82	$82 - 84 = -2$
95	$95 - 84 = +11$
67	$67 - 84 = -17$
92	$92 - 84 = +8$
	$\Sigma(x - \bar{x}) = 0$

[2]Note that Σ is uppercase sigma and σ is lowercase sigma of the Greek alphabet.

[3]From the formula for σ^2, it can be stated that the population variance is the mean of the squared deviations of x values from the mean. However, this is not true for the variance calculated for a sample data set.

The population variance is the average of the squares of the distance each value is from the mean.

❖ The standard deviation is the square root of the variance.

Uses of the Variance and Standard Deviation

- ✓ To determine the spread of the data.
- ✓ To determine the consistency of a variable.
- ✓ To determine the number of data values that fall within a specified
- ✓ interval in a distribution.
- ✓ Used in inferential statistics.

- The population variance is

$$\sigma^2 = \frac{\sum (X - \mu)^2}{N}$$

- The **population standard deviation** is

$$\sigma = \sqrt{\frac{\sum (X - \mu)^2}{N}}$$

Step 1 Find the mean for the data. $\mu = \frac{\Sigma X}{N}$

Step 2 Find the Deviation for each data value. $X - \mu$ انحراف

Step 3 Square each of the deviations. $(X - \mu)^2$

Step 4 Find the sum of the squares. $\Sigma(X - \mu)^2$

Example

Find the variance and standard deviation for the data set for Brand A paint. 10, 60, 50, 30, 40, 20

Months, X	μ	$X-\mu$	$(X-\mu)^2$
10	35	−25	625
60	35	25	625
50	35	15	225
30	35	−5	25
40	35	5	25
20	35	−15	225
			1750

$$\sigma^2 = \frac{\sum (X-\mu)^2}{n} = \frac{1750}{6} = \boxed{291.7}$$

$$\sigma = \sqrt{\frac{1750}{6}} = \boxed{17.1}$$

- The **sample variance** is

$$s^2 = \frac{\sum (X-\bar{X})^2}{n-1}$$

- The **sample standard deviation** is

$$s = \sqrt{\frac{\sum (X-\bar{X})^2}{n-1}}$$

Coefficient of Variation

The ***Coefficient of variation*** is the standard deviation divided by the mean, expressed as a percentage.

$$CVAR = \frac{s}{X} .100$$

Use *CVAR* to compare standard deviations when the units are different.

Example: Sales of Automobiles

The mean of the number of sales of cars over a 3-month period is 87, and the standard deviation is 5. The mean of the commissions is $5225, and the standard deviation is $773. Compare the variations of the two

CVAR=$\frac{5}{78}$ .100=5.7% Sales

CVAR=$\frac{773}{5225}$ .100=14.8 % Commissions

CHAPTER 2

Probability

1. Introduction

A cynical person once said, “The only two sure things are death and taxes.” This philosophy no doubt arose because so much in people’s lives is affected by chance. From the time you awake until you go to bed, you make decisions regarding the possible events that are governed at least in part by chance. For example, should you carry an umbrella to work today? Will your car battery last until spring? Should you accept that new job?

Probability as a general concept can be defined as the chance of an event occurring. Many people are familiar with probability from observing or playing games of chance, such as card games, slot machines, or lotteries. In addition to being used in games of chance, probability theory is used in the fields of insurance, investments, and weather forecasting and in various other areas. Finally, as stated in Chapter 1, probability is the basis. of inferential statistics. For example, predictions are based on probability, and hypotheses are tested by using probability.

The basic concepts of probability are explained in this chapter. These concepts include probability experiments, sample spaces, the addition and multiplication rules, and the probabilities of complementary events. Also in this chapter, you will learn the rule for counting, the differences between permutations and combinations, and how to figure out how many different combinations for specific situations exist. Finally, Section1–5 explains how the counting rules and the probability rules can be used together to solve a wide variety of problems.

2. Sample Spaces and Probability

The theory of probability grew out of the study of various games of chance using coins, dice, and cards. Since these devices lend themselves well to the application of concepts of probability, they will be used in this chapter as examples. This section begins by explaining some basic concepts of probability. Then the types of probability and probability rules are discussed.

2 Basic Concepts

Processes such as flipping a coin, rolling a die, or drawing a card from a deck are called **probability experiments**.

***probability experiment*: is a chance process that leads to well-defined results called outcomes.**

An *outcome* is the result of a single trial of a probability experiment.

A trial means flipping a coin once, rolling one die once, or the like. When a coin is tossed, there are two possible outcomes: head or tail. (*Note:* We exclude the possibility of a coin landing on its edge.) In the roll of a single die, there are six possible outcomes: 1, 2, 3, 4, 5, or 6. In

any experiment, the set of all possible outcomes is called the *sample space.*

A sample space is the set of all possible outcomes of a probability experiment.

Some sample spaces for various probability experiments are shown here.

Experiment	Experiment
Toss one coin	**Head, tail**
Roll a die	**1, 2, 3, 4, 5, 6**
Answer a true/false question	**True, false**
Toss two coins	**Head-head, tail-tail, head-tail, tail-head**

It is important to realize that when two coins are tossed, there are *four* possible outcomes, as shown in the fourth experiment above. Both coins could fall heads up. Both coins could fall tails up. Coin 1 could fall heads up and coin 2 tails up. Or coin 1 could fall tails up and coin 2 heads up. Heads and tails will be abbreviated as H and T throughout this chapter.

***Example 1*: Find the sample space for rolling two dice**

Solution

Since each die can land in six different ways, and two dice are rolled, the sample space can be presented by a rectangular array, as shown in Figure below . The sample space is the list of pairs of numbers in the chart.

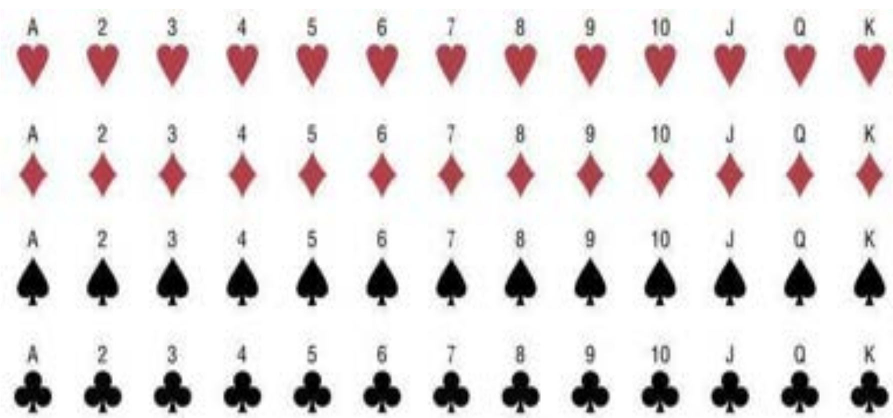
Die 1	Die 2					
	1	2	3	4	5	6
1	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
2	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
3	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
4	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
5	(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
6	(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

Example 2:

Find the sample space for drawing one card from an ordinary deck of cards.

Solution

Since there are 4 suits (hearts, clubs, diamonds, and spades) and 13 cards for each suit (ace through king), there are 52 outcomes in the sample space.



***Example 3*:**

Find the sample space for the gender of the children if a family has three children. Use B for boy and G for girl.

Solution

There are two genders, male and female, and each child could be either gender. Hence, there are eight possibilities, as shown here.

BBB BBG BGB GBB GGG GGB GBG BGG

- **In Examples 1–1 through 1–3, the sample spaces were found by observation and reasoning; however, another way to find all possible outcomes of a probability experiment is to use a tree diagram.**
- **The sample space for an experiment can also be illustrated by drawing either a Venn diagram or a tree diagram. A *Venn diagram* is a picture (a closed geometric shape such as a rectangle, a square, or a circle) that depicts all the possible outcomes for an experiment. In a *tree diagram*, each outcome is represented by a branch of the tree. Venn and tree diagrams help us understand probability concepts by presenting them visually,**

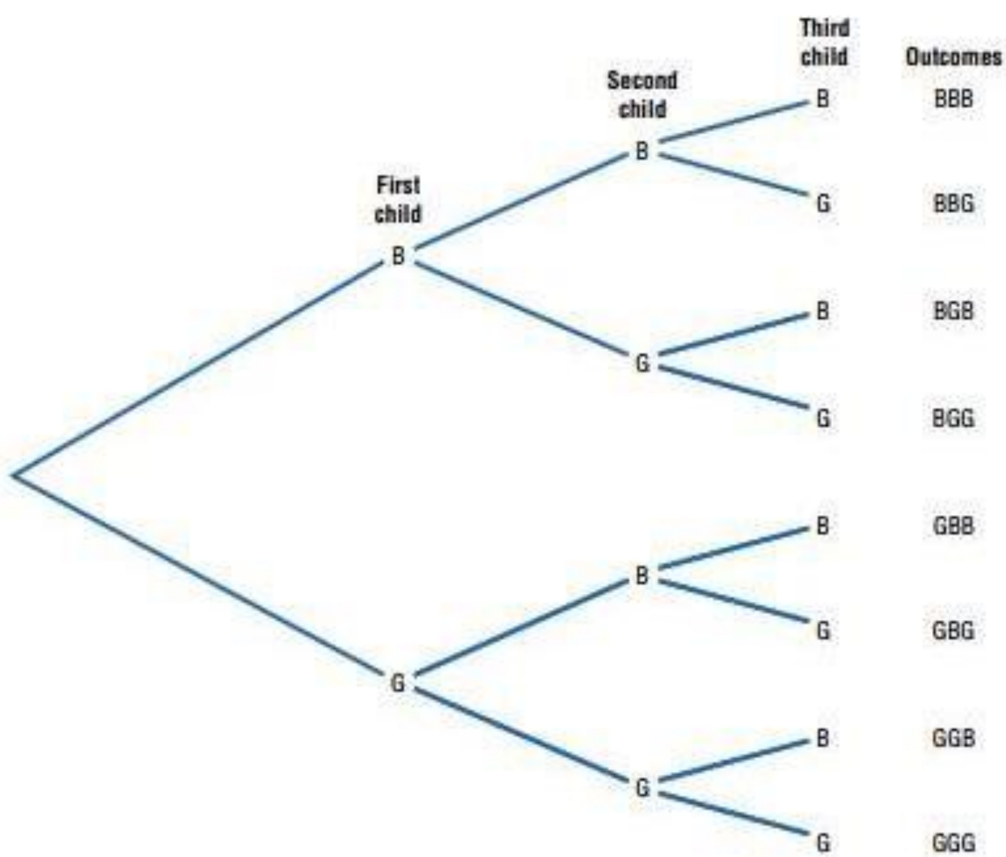
Tree diagram **is a device consisting of line segments emanating from a starting point and also from the outcome point. It is used to determine all possible outcomes of a probability experiment.**

Example **4**

Use a tree diagram to find the sample space for the gender of three children in a family

Solution

Since there are two possibilities (boy or girl) for the first child, draw two branches from a starting point and label one B and the other G. Then if the first child is a boy, there are two possibilities for the second child (boy or girl), so draw two branches from B and label one B and the other G. Do the same if the first child is a girl. Follow the same procedure for the third child. The completed tree diagram is shown in Figure Below . To find the outcomes for the sample space, trace through all the possible branches, beginning at the starting point for each one.



An outcome was defined previously as the result of a single trial of a probability experiment. In many problems, one must find the probability of two or more outcomes. For this reason, it is necessary to distinguish between an outcome and an event.

Event **consists of a set of outcomes of a probability experiment.**

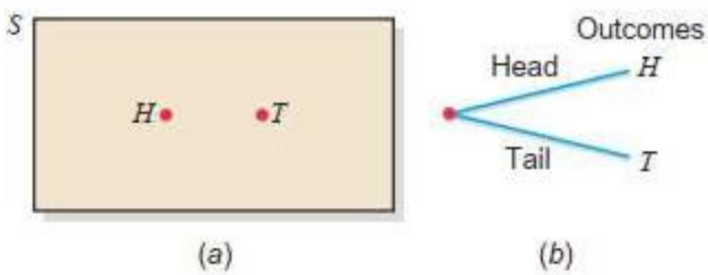
An event can be one outcome or more than one outcome. For example, if a die is rolled and a 6 shows, this result is called an outcome, since it is a result of a single trial. An event with one outcome is called a simple event**. The event of getting an odd number. when a die is rolled is called a** compound event**, since it consists of three outcomes or three simple events. In general, a compound event consists of two or more outcomes or simple events.**

Example 5

Draw the Venn and tree diagrams for the experiment of tossing a coin once.

This experiment has two possible outcomes: head and tail. Consequently, the sample space is given by S={H,T}, Where H= head ,T= Tail

To draw a Venn diagram for this example, we draw a rectangle and mark two points inside this rectangle that represent the two outcomes, Head and Tail. The rectangle is labeled *S* because it represents the sample space (see Figure). To draw a tree diagram, we draw two branches starting at the same point, one representing the head and the second representing the tail. The two final outcomes are listed at the ends of the branches



Example 6 : **Draw the Venn and tree diagrams for the experiment of tossing a coin twice.**

S={HH,TT,HT,TH}.

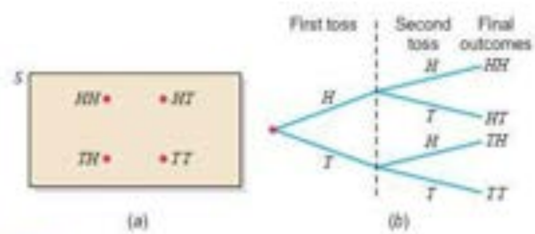
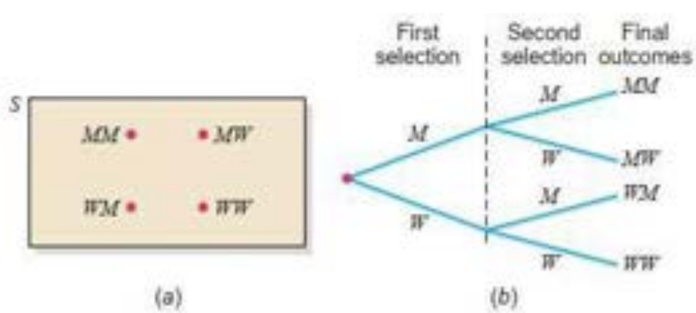


Figure (a) Venn diagram and (b) tree diagram for two tosses of a coin.

Example 6

Suppose we randomly select two workers from a company and observe whether the worker selected each time is a man or a woman. Write all the outcomes for this experiment. Draw the Venn and tree diagrams for this experiment



3. probability theorems

Introduction

We often make statements about probability. For example, a weather forecaster may predict that there is an 80% chance of rain tomorrow. A health news reporter may state that a smoker has a much greater chance of getting cancer than does a nonsmoker. A college student may ask an instructor about the chances of passing a course or getting an A if he or she did not do well on the midterm examination. Probability, which measures the likelihood that an event will occur, is an important part of statistics. It is the basis of inferential statistics, which will be introduced in later chapters. In inferential statistics, we make decisions under conditions of uncertainty. Probability theory is used to evaluate the uncertainty involved in those decisions. For example, estimating next year's sales for a company is based on many assumptions, some of which may happen to be true and others may not. Probability theory will help us make decisions under such conditions of imperfect information and uncertainty. Combining probability and probability distributions with descriptive statistics will help us make decisions about populations based on information obtained from samples. This chapter presents the basic concepts of probability and the rules for computing probability.

There are three basic interpretations of probability:

1. Classical probability

2. Empirical or relative frequency probability

3. Subjective probability

1- *Classical Probability*

Classical probability uses sample spaces to determine the numerical probability that an event will happen. You do not actually have to perform the experiment to determine that probability. Classical probability is so named because it was the first type of probability studied formally by mathematicians in the 17th and 18th centuries .Classical probability assumes that all outcomes in the sample space are equally likely to occur. For example, when a single die is rolled, each outcome has the same probability of occurring. Since there are six outcomes, each outcome has a probability of (*1/6*).When a card is selected from an ordinary deck of 52 cards, you assume that the deck has been shuffled, and each card has the same probability of being selected. In this case, it is (*1/52*).

Equally likely events are events that have the same probability of occurring.

Formula for Classical Probability

The probability of any event E is

$$\frac{\text{Number of outcomes in } E}{\text{Total number of outcomes in the sample space}}$$

This probability is denoted by

$$P(E) = \frac{n(E)}{n(S)}$$

This probability is called *classical probability*, and it uses the sample space S.

Probabilities can be expressed as fractions, decimals, or—where appropriate— percentages. If one asks, “What is the probability of getting a head when a coin is tossed?” typical responses can be any of the following three.

“One-half.”, “Point five.”, “Fifty percent.”1

These answers are all equivalent. In most cases, the answers to examples and exercises given in this chapter are expressed as fractions or decimals, but percentages are used where appropriate.

Rounding Rule for Probabilities *Probabilities should be expressed as reduced fractions or rounded to two or three decimal places. When the probability of an event is an extremely small decimal, it is permissible to round the decimal to the first nonzero digit after the point. For example, 0.0000587 would be 0.00006. When obtaining probabilities from one of the tables in Appendix C, use the number of decimal places given in the table. If decimals are converted to percentages to express probabilities, move the decimal point two places to the right and add a percent sign.*

Example7

Find the probability of getting a black 10 when drawing a card from a deck.

Solution / There are 52 cards in a deck, and there are two black 10s—the 10 of spades and the 10 of clubs. Hence the

probability of getting a black 10 is P(black 10)= $\frac{2}{52} = \frac{1}{26}$.

Example 8

If a family has three children, find the probability that two of the three children are girls. Solution

The sample space for the gender of the children for a family that has three children has eight outcomes, that is, BBB, BBG, BGB, GBB, GGG, GGB, GBG, and BGG.

Since there are three ways to have two girls, namely, GGB, GBG, and BGG, P(two girls)= $\frac{3}{8}$

Remark

In probability theory, it is important to understand the meaning of the words *and* .*and* *or.* For example, if you were asked to find the probability of getting a queen *and* a heart when you are drawing a single card from a deck, you would be looking for the queen of hearts. Here the word *and* means “at the same time.” The word *or* has two meanings. For example, if you were asked to find the probability of selecting a queen *or* a heart when one card is selected from a deck, you would be looking for one of the 4 queens or one of the 13 hearts. In this case, the queen of hearts would be included in both cases and counted twice. So there would be

$4 + 13 - 1 = 16$ possibilities.

On the other hand, if you were asked to find the probability of getting a queen *or* a king, you would be looking for one of the 4 queens or one of the 4 kings. In this case, there would be $4 + 4 = 8$ possibilities. In the first case, both events can occur at the same time; we say that this is an example of the *inclusive or.* In the second case, both events cannot occur at the same time, and we say that this is an example of the *exclusive or.*

Example 9

A card is drawn from an ordinary deck. Find these probabilities.

a. Of getting a jack

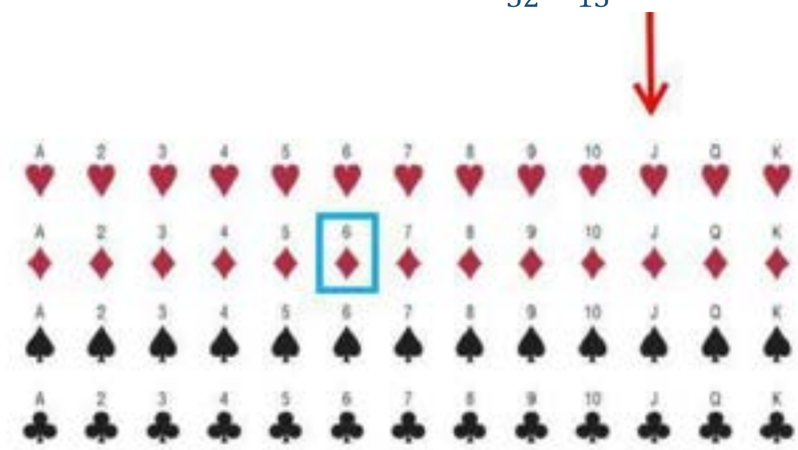
b. Of getting the 6 of clubs (i.e., a 6 and a club)

c. Of getting a 3 or a diamond

d. Of getting a 3 or a 6.

solution

a. **Refer to the sample space in Figure 1–2. There are 4 jacks so there are 4 outcomes in event E and 52 possible outcomes in the sample space. Hence, P(jack) =** $\frac{4}{52} = \frac{1}{13}$



b. *Since there is only one 6 of clubs in event* E, *the probability of getting a 6 of clubs is* P(6 of clubs) = $\frac{1}{52}$.

c. **There are four 3s and 13 diamonds, but the 3 of diamonds is counted twice in this listing. Hence, there are 16 possibilities of drawing a 3 or a diamond, so.**

P(3 or diamond) = $\frac{16}{52} = \frac{4}{13}$.

This is an example of the inclusive or

d. **Since there are four 3s and four 6s,**

***P*(3 or 6) =** $\frac{8}{52} = \frac{2}{13}$

This is an example of the exclusive or.

There are four basic probability rules. These rules are helpful in solving probability problems, in understanding the nature of probability, and in deciding if your answers to the problems are correct.

Probability Rule 1

The probability of any event E is a number (either a fraction or decimal) between and including 0 and 1. This is denoted by $0 \leq P(E) \leq 1$.

Rule 1 states that probabilities cannot be negative or greater than 1

Probability Rule 2

If an event E cannot occur (i.e., the event contains no members in the sample space), its probability is 0.

Example 10

When a single die is rolled, find the probability of getting a 9.

Solution

Since the sample space is 1, 2, 3, 4, 5, and 6, it is impossible to get a 9. Hence, the probability is $P(9) = \frac{0}{6} = 6$

Probability Rule 3

If an event E is certain, then the probability of E is 1.

In other words, if $P(E)$ 5 1, then the event E is certain to occur. This rule is illustrated in Example 11.

Example11.

When a single die is rolled, what is the probability of getting a number less than 7?

Solution

Since all outcomes—1, 2, 3, 4, 5, and 6—are less than 7, the probability is *P*(number less than 7) =$\frac{6}{6}$=1.

The event of getting a number less than 7 is certain.

In other words, probability values range from 0 to 1. When the probability of an event is close to 0, its occurrence is highly unlikely. When the probability of an event is near 0.5, there is about a 50-50 chance that the event will occur; and when the probability of an event is close to 1, the event is highly likely to occur.

Probability Rule 4

The sum of the probabilities of all the outcomes in the sample space is 1.

For example, in the roll of a fair die, each outcome in the sample space has a probability of . Hence, the sum of the probabilities of the outcomes is as shown.

Outcome	1		2		3		4		5		6
Probability	$\frac{1}{6}$		$\frac{1}{6}$		$\frac{1}{6}$		$\frac{1}{6}$		$\frac{1}{6}$		$\frac{1}{6}$
Sum	$\frac{1}{6}$	+	$\frac{1}{6}$	+	$\frac{1}{6}$	+	$\frac{1}{6}$	+	$\frac{1}{6}$	+	$\frac{1}{6} = \frac{6}{6} = 1$

Complementary Events

Another important concept in probability theory is that of complementary events. *When a die is rolled, for instance, the sample space consists of the* *outcomes 1, 2, 3, 4, 5, and 6. The event* E *of getting odd numbers consists of the outcomes 1, 3, and 5. The event of not getting an odd number is called the* complement *of event* E, *and it consists of the outcomes 2, 4, and 6.*

The *complement of an event* E is the set of outcomes in the sample space that are not included in the outcomes of event E. The complement of E is denoted by (read "E bar").

Example 11

Find the complement of each event. *a*. **Rolling a die and getting a 4**

b. **Selecting a letter of the alphabet and getting a vowel**

c. **Selecting a month and getting a month that begins with a J**

d. **Selecting a day of the week and getting a weekday.**

Solution

a. *Getting a 1, 2, 3, 5, or 6*

b. *Getting a consonant (assume* y *is a consonant)*

c. *Getting February, March, April, May, August, September, October, November,*

or December

d. *Getting Saturday or Sunday*

The outcomes of an event and the outcomes of the complement make up the entire sample space. For example, if two coins are tossed, the sample space is HH, HT, TH, and TT. The complement of "getting all heads" is not "getting all tails," since the event "all heads" is HH, and the complement of HH is HT, TH, and TT. Hence, the complement of the event "all heads" is the event "getting at least one tail."

Since the event and its complement make up the entire sample space, it follows that the sum of the probability of the event and the probability of its complement.

will equal 1. That is, $P(E) + P(\overline{E}) = 1$. In Example 4–10, let E = all heads, or HH, and let $\overline{E}$ = at least one tail, or HT, TH, TT. Then $P(E) = \frac{1}{4}$ and $P(\overline{E}) = \frac{3}{4}$; hence, $P(E) + P(\overline{E}) = \frac{1}{4} + \frac{3}{4} = 1$.

The rule for complementary events can be stated algebraically in three ways.

Rule for Complementary Events

$$P(\overline{E}) = 1 - P(E) \quad \text{or} \quad P(E) = 1 - P(\overline{E}) \quad \text{or} \quad P(E) + P(\overline{E}) = 1$$

Stated in words, the rule is: *If the probability of an event or the probability of its complement is known, then the other can be found by subtracting the probability from 1*. **This rule is important in probability theory because at times the best solution to a problem is to find the probability of the complement of an event and then subtract from 1 to get the probability of the event itself.**

Example 12

If the probability that a person lives in an industrialized country of the world is , find the probability that a person does not live in an industrialized country.

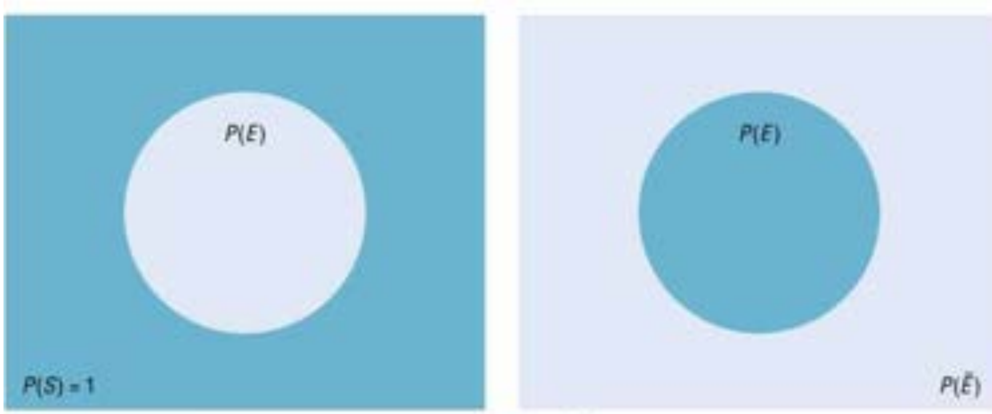
Solution

P*(not living in an industrialized country)* =*1* - P*(living in an industrialized country)*

$$=1-\frac{1}{5}=\frac{4}{5}.$$

Probabilities can be represented pictorially by *Venn diagrams.* Figure 4–4(a) shows the probability of a simple event E. The area inside the circle represents the probability of event E, that is, P(E). The area inside the rectangle represents the probability of all the events in the sample space P(S).

The Venn diagram that represents the probability of the complement of an event $P(\overline{E})$ is shown in Figure below In this case, $P(\overline{E}) = 1 - P(E)$, which is the area inside the rectangle but outside the circle representing $P(E)$. Recall that $P(S) = 1$ and $P(E) = 1 - P(\overline{E})$. The reasoning is that $P(E)$ is represented by the area of the circle and $P(\overline{E})$ is the probability of the events that are outside the circle.



(a) Simple probability (b) $P(\bar{E}) = 1 - P(E)$

2- Empirical Probability

The difference between classical and *empirical probability* is that classical probability assumes that certain outcomes are equally likely (such as the outcomes when a die is rolled), while empirical probability

relies on actual experience to determine the likelihood of outcomes. In empirical probability, one might actually roll a given die 6000 times, observe the various frequencies, and use these frequencies to determine the probability of an outcome. Suppose, for example, that a researcher for the American Automobile Association (AAA) asked 50 people who plan to travel over the Thanksgiving holiday how they will get to their destination. The results can be categorized in a frequency distribution as shown.

Method	Frequency
Drive	41
Fly	6
Train or bus	3
	50

Now probabilities can be computed for various categories. For example, the probability of selecting a person who is driving is =$\frac{41}{50}$ since 41 out of the 50 people said that they were driving

Formula for Empirical Probability

Given a frequency distribution, the probability of an event being in a given class is

$$P(E) = \frac{\text{frequency for the class}}{\text{total frequencies in the distribution}} = \frac{f}{n}$$

This probability is called *empirical probability* and is based on observation.

Example 13

In the travel survey just described, find the probability that a person will travel by airplane over the Thanksgiving holiday.

Solution

$$P(E) = \frac{f}{n} = \frac{6}{50} = \frac{3}{25}$$

Example 13

In a sample of 50 people, 21 had type O blood, 22 had type A blood, 5 had type B blood, and 2 had type AB blood. Set up a frequency distribution and find the following probabilities.

a. A person has type O blood.

b. A person has type A or type B blood.

c. A person has neither type A nor type O blood.

d. A person does not have type AB blood.

Solution

Type	frequency
A	**22**
B	**5**
AB	2
O	21
Total	**50**

a. $P(\text{O}) = \frac{f}{n} = \frac{21}{50}$

b. $P(\text{A or B}) = \frac{22}{50} + \frac{5}{50} = \frac{27}{50}$

(Add the frequencies of the two classes.)

c. $P(\text{neither A nor O}) = \frac{5}{50} + \frac{2}{50} = \frac{7}{50}$

(Neither A nor O means that a person has either type B or type AB blood.)

d. $P(\text{not AB}) = 1 - P(\text{AB}) = 1 - \frac{2}{50} = \frac{48}{50} = \frac{24}{25}$

(Find the probability of not AB by subtracting the probability of type AB from 1.)

Example 14

Hospital records indicated that maternity patients stayed in the hospital for the number of days shown in the distribution.

Number of days stayed	Number of days stayed
3	**15**
4	**32**
5	**56**
6	**19**
7	**5**
	127

Find these probabilities.

a. **A patient stayed exactly 5 days.** *c.* **A patient stayed at most 4 days.**

b. **A patient stayed less than 6 days.** *d.* **A patient stayed at least 5 days.**

Solution

a. $P(5) = \frac{56}{127}$

b. $P(\text{less than 6 days}) = \frac{15}{127} + \frac{32}{127} + \frac{56}{127} = \frac{103}{127}$

(Less than 6 days means 3, 4, or 5 days.)

c. $P(\text{at most 4 days}) = \frac{15}{127} + \frac{32}{127} = \frac{47}{127}$

(At most 4 days means 3 or 4 days.)

d. $P(\text{at least 5 days}) = \frac{56}{127} + \frac{19}{127} + \frac{5}{127} = \frac{80}{127}$

(At least 5 days means 5, 6, or 7 days.)

3- Subjective Probability

The third type of probability is called subjective probability. Subjective probability uses a probability value based on an educated guess or estimate, employing opinions and inexact information.

In subjective probability, a person or group makes an educated guess at the chance that an event will occur. This guess is based on the person's experience and evaluation of a solution. For example, a sportswriter may say that there is a 70% probability that the Pirates will win the pennant next year. A physician might say that, on the basis of her diagnosis, there is a 30% chance the patient will need an operation. A seismologist might say there is an 80% probability that an earthquake will occur in a certain area. These are only a few examples of how subjective probability is used in everyday life.

All three types of probability (classical, empirical, and subjective) are used to solve a variety of problems in business, engineering, and other fields.

4. The Addition Rules for Probability

Many problems involve finding the probability of two or more events. For example, at a large political gathering, one might wish to know, for a person selected at random, the probability that the person is a female or is a Republican. In this case, there are three possibilities to consider:

1. **The person is a female.**

2. **The person is a Republican.**

3. **The person is both a female and a Republican.**

Consider another example. At the same gathering there are Republicans, Democrats, and Independents. If a person is selected at random, what is the probability that the person is a Democrat or an Independent? In this case, there are only two possibilities:

1. **The person is a Democrat.**

2. **The person is an Independent.**

The difference between the two examples is that in the first case, the person selected can be a female and a Republican at the same time. In the second case, the person selected cannot be both a Democrat and an Independent at the same time. In the second case, the two events are said to be mutually exclusive; **in the first case, they are not mutually exclusive**

Two events are *mutually exclusive events* if they cannot occur at the same time (i.e., they have no outcomes in common)

In another situation, the events of getting a 4 and getting a 6 when a single card is drawn from a deck are mutually exclusive events, since a single card cannot be both a 4 and a 6. On the other hand, the events of getting a 4 and getting a heart on a single draw are not mutually exclusive, since one can select the 4 of hearts when drawing a single card from an ordinary deck

Example 15

Determine which events are mutually exclusive and which are not, when a single die is rolled.

a. **Getting an odd number and getting an even number**

b. **Getting a 3 and getting an odd number**

c. **Getting an odd number and getting a number less than 4**

d. **Getting a number greater than 4 and getting a number less than 4**

Solution

a. *The events are mutually exclusive, since the first event can be 1, 3, or 5 and the*

second event can be 2, 4, or 6.

b. *The events are not mutually exclusive, since the first event is a 3 and the second*

can be 1, 3, or 5. Hence, 3 is contained in both events.

c. *The events are not mutually exclusive, since the first event can be 1, 3, or 5 and*

the second can be 1, 2, or 3. Hence, 1 and 3 are contained in both events.

d. *The events are mutually exclusive, since the first event can be 5 or 6 and the* *second event can be 1, 2, or 3.*

Example 16

Determine which events are mutually exclusive and which are not, when a single card is drawn from a deck.

a. **Getting a 7 and getting a jack**

b. **Getting a club and getting a king**

c. **Getting a face card and getting an ace**

d. **Getting a face card and getting a spade**

solution

Only the events in parts *a* and *c* are mutually exclusive.

The probability of two or more events can be determined by the *addition rules.* The first addition rule is used when the events are mutually exclusive.

Addition Rule 1

When two events A and B are mutually exclusive, the probability that A or B will occur is

$$P(A \text{ or } B) = P(A) + P(B)$$

Example 17

A city has 9 coffee shops: 3 Starbuck's, 2 Caribou Coffees, and 4 Crazy Mocho Coffees. If a person selects one shop at random to buy a cup of

coffee, find the probability that it is either a Starbuck's or Crazy Mocho Coffees.

<u>Solution</u>

Since there are 3 Starbuck's and 4 Crazy Mochos, and a total of 9 coffee shops, P(Starbuck's or Crazy Mocho)

= P(Starbuck's) + P(Crazy Mocho)= $\frac{3}{9}+\frac{4}{9}=\frac{7}{9}$

Example 18

The corporate research and development centers for three local companies have the following number of employees:

U.S. Steel 110

Alcoa 750

Bayer Material Science 250

If a research employee is selected at random, find the probability that the employee is employed by U.S. Steel or Alcoa.

<u>Solution</u>

$$P(\text{U.S. Steel or Alcoa}) = P(\text{U.S. Steel}) + P(\text{Alcoa}) = \frac{110}{1110}+\frac{750}{1110}=\frac{860}{1110}=\frac{86}{111}$$

Example 19

A day of the week is selected at random. Find the probability that it is a weekend day.

<u>Solution</u>

P(Saturday or Sunday) = P(Saturday) + P(Sunday)= $\frac{1}{7}+\frac{1}{7}=\frac{2}{7}$

When two events are not mutually exclusive, we must subtract one of the two probabilities of the outcomes that are common to both events, since they have been counted twice. This technique is illustrated in Example 20.

Example 20

A single card is drawn at random from an ordinary deck of cards. Find the probability that it is either an ace or a black card.

Solution

Since there are 4 aces and 26 black cards (13 spades and 13 clubs), 2 of the aces are black cards, namely, the ace of spades and the ace of clubs. Hence the probabilities of the two outcomes must be subtracted since they have been counted twice.

P(ace or black card) = P(ace) + P(black card) - P(black aces) = $\frac{4}{52}+\frac{26}{52}-\frac{2}{52}=\frac{28}{52}=\frac{7}{13}$.

When events are not mutually exclusive, addition rule 2 can be used to find the probability of the events.

Addition Rule 2

If A and B are *not* mutually exclusive, then

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

Note:

This rule can also be used when the events are mutually exclusive, since P(A and B) will always equal 0. However, it is important to make a distinction between the two situations

Example 21

In a hospital unit there are 8 nurses and 5 physicians; 7 nurses and 3 physicians are females. If a staff person is selected, find the probability that the subject is a nurse or a male.

Solution

The sample space is shown here.

Staff	Females	Males	Total
Nurses	7	1	8
Physicians	3	2	5
Total	10	3	13

The probability is

$$P(\text{nurse or male}) = P(\text{nurse}) + P(\text{male}) - P(\text{male nurse})$$
$$= \tfrac{8}{13} + \tfrac{3}{13} - \tfrac{1}{13} = \tfrac{10}{13}$$

Example 22.

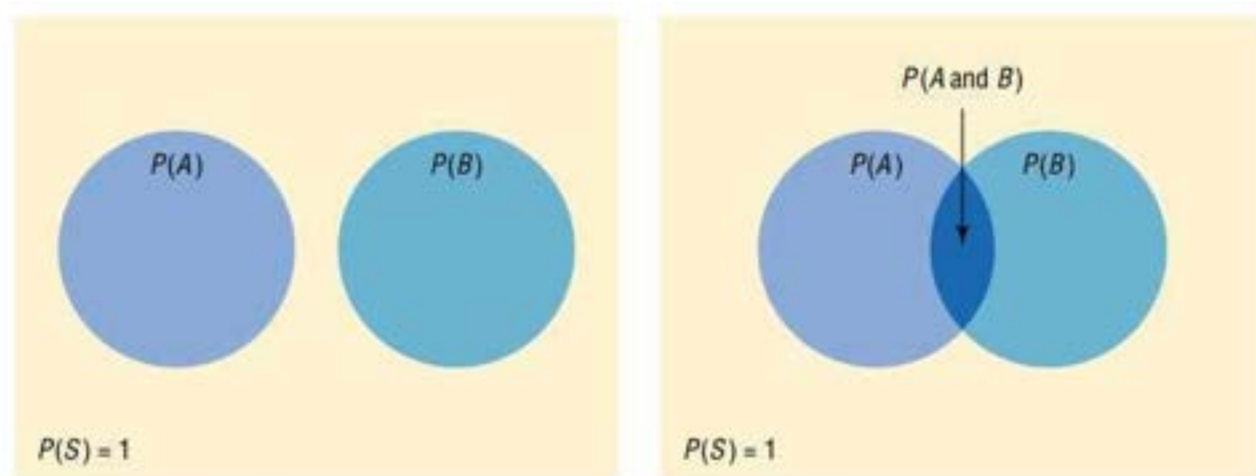
On New Year's Eve, the probability of a person driving while intoxicated is 0.32, the probability of a person having a driving accident is 0.09, and the probability of a person having a driving

accident while intoxicated is 0.06. What is the probability of a person driving while intoxicated or having a driving accident?

<u>Solution</u>

P(intoxicated or accident) = P(intoxicated) + P(accident) - P(intoxicated and accident) = 0.32 +0.09 - 0.06 = 0.35.

In summary, then, when the two events are mutually exclusive, use addition rule 1. When the events are not mutually exclusive, use addition rule 2. The probability rules can be extended to three or more events. For three mutually exclusive events A, B, and C, P(A or B or C) = P(A) + P(B) + P(C).



(a) Mutually exclusive events
P(A or B) = P(A) + P(B)

(b) Non-mutually exclusive events
P(A or B) = P(A) + P(B) − P(A and B)

For three events that are not mutually exclusive .

P(A or B or C) = P(A) + P(B) - P(C) - P(A and B) - P(A and C) - P(B and C) + P(A and B and C).

Above figure (a) shows a Venn diagram that represents two mutually exclusive events A and B. In this case, P(A or B) = P(A) + P(B), since these events are mutually exclusive and do not overlap. In other words,

the probability of occurrence of event A or event B is the sum of the areas of the two circles.

Figure (b) represents the probability of two events that are not mutually exclusive. In this case, P(A or B) = P(A) + P(B) - P(A and B). The area in the intersection or overlapping part of both circles corresponds to P(A and B); and when the area of circle A is added to the area of circle B, the overlapping part is counted twice. It must therefore be subtracted once to get the correct area or probability.

5. The Multiplication Rules and Conditional Probability

The multiplication rules can be used to find the probability of two or more events that occur in sequence. For example, if you toss a coin and then roll a die, you can find the probability of getting a head on the coin and a 4 on the die. These two events are said to be independent since the outcome of the first event (tossing a coin) does not affect the probability outcome of the second event (rolling a die).

Two events A and B are independent events if the fact that A occurs does not affect the probability of B occurring.

Here are other examples of independent events:

Rolling a die and getting a 6, and then rolling a second die and getting a 3. Drawing a card from a deck and getting a queen, replacing it, and drawing a second card and getting a queen. To find the probability of two independent events that occur in sequence, you must find the probability of each event occurring separately and then multiply the

answers. For example, if a coin is tossed twice, the probability of getting two heads is

$\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$ This result can be verified by looking at the sample space HH, HT, TH, TT. Then P(HH) = $\frac{1}{4}$

Multiplication Rule 1

When two events are independent, the probability of both occurring is

$$P(A \text{ and } B) = P(A) \cdot P(B)$$

Example 23

A coin is flipped and a die is rolled. Find the probability of getting a head on the coin and a 4 on the die.

Solution

P(head and 4) =P(head) * P(4) = $\frac{1}{2} * \frac{1}{6} = \frac{1}{12}$

Note that the sample space for the coin is H, T; and for the die it is 1, 2, 3, 4, 5, 6. The problem in Example 23 can also be solved by using the sample space

H1 H2 H3 H4 H5 H6 T1 T2 T3 T4 T5 T6 The solution is 1 , since there is only one way to get the head-4 outcome

Example 24

A card is drawn from a deck and replaced; then a second card is drawn. Find the probability of getting a queen and then an ace.

Solution

The probability of getting a queen is , and since the card is replaced, the probability of getting an ace is . Hence, the probability of getting a queen and an ace is

P(queen and ace) = P(queen) . P(ace)= $\frac{4}{52} * \frac{4}{52} = \frac{16}{2704} = \frac{1}{169}$

Example 25:

An urn contains 3 red balls, 2 blue balls, and 5 white balls. A ball is selected and its color noted. Then it is replaced. A second ball is selected and its color noted. Find the probability of each of these.

a. Selecting 2 blue balls

b. Selecting 1 blue ball and then 1 white ball

c. Selecting 1 red ball and then 1 blue ball

a. $P(\text{blue and blue}) = P(\text{blue}) \cdot P(\text{blue}) = \frac{2}{10} \cdot \frac{2}{10} = \frac{4}{100} = \frac{1}{25}$

b. $P(\text{blue and white}) = P(\text{blue}) \cdot P(\text{white}) = \frac{2}{10} \cdot \frac{5}{10} = \frac{10}{100} = \frac{1}{10}$

c. $P(\text{red and blue}) = P(\text{red}) \cdot P(\text{blue}) = \frac{3}{10} \cdot \frac{2}{10} = \frac{6}{100} = \frac{3}{50}$

Multiplication rule 1 can be extended to three or more independent events by using the formula

P(A and B and C and . . . and K) = P(A) . P(B) . P(C) . . . P(K)

Note

When a small sample is selected from a large population **and the subjects are not replaced, the probability of the event occurring changes so slightly that for the most part, it is considered to remain the same. Examples 4–26 and 4–27 illustrate this concept**

Example 26

A Harris poll found that 46% of Americans say they suffer great stress at least once a week. If three people are selected at random, find the probability that all three will say that they suffer great stress at least once a week.

Solution

Let S denote stress. Then

$P(S \text{ and } S \text{ and } S)$ 5 $P(S) \bullet P(S) \bullet P(S)$

(0.46)(0.46)*(0.46) = 0.097*

Example 27

Approximately 9% of men have a type of color blindness that prevents them from distinguishing between red and green. If 3 men are selected at random, find the probability that all of them will have this type of red-green color blindness.

Solution

Let C denote red-green color blindness. Then $P(C \text{ and } C \text{ and } C) = P(C) \bullet P(C) \bullet P(C)$

= (0.09)*(0.09)*(0.09)

= 0.000729 Hence, the rounded probability is 0.0007.

the events were independent of one another, since the occurrence of the first event in no way affected the outcome of the second event. On the other hand, when the occurrence of the first event changes the probability of the occurrence of the second event, the two events are said to be dependent. For example, suppose a card is drawn from a deck and not replaced, and then a second card is drawn. What is the probability of selecting an ace on the first card and a king on the second card?

Before an answer to the question can be given, you must realize that the events are dependent. The probability of selecting an ace on the first draw is . If that card is not replaced, the probability of selecting a king on the second card is , since there are 4 kings and 51 cards remaining. The outcome of the first draw has affected the outcome of the second draw. Dependent events are formally defined now

When the outcome or occurrence of the first event affects the outcome or occurrence of the second event in such a way that the probability is changed, the events are said to be **dependent events.**

Here are some examples of dependent events:

Drawing a card from a deck, not replacing it, and then drawing a second card.

Selecting a ball from an urn, not replacing it, and then selecting a second ball.

Being a lifeguard and getting a suntan.

Having high grades and getting a scholarship.

Parking in a no-parking zone and getting a parking ticket.

To find probabilities when events are dependent, use the multiplication rule with a modification in notation. For the problem just discussed, the probability of getting an ace on the first draw is , and the probability of getting a king on the second draw is . By the multiplication rule, the probability of both events occurring is

$$\frac{4}{52} * \frac{4}{51} = \frac{16}{2652} = \frac{4}{663}$$

The event of getting a king on the second draw given that an ace was drawn the first time is called a conditional probability. The conditional probability of an event B in relationship to an event A is the probability that event B occurs after event A has already occurred.
The notation for conditional

probability is P(B|A). This notation does not mean that B is divided by A; rather, it means the probability that event B occurs given that event A has already occurred. In the card example, P(B|A) is the probability that the second card is a king given that the first card is an ace, and it is equal to 4/51 since the first card was not replaced.

Multiplication Rule 2

When two events are dependent, the probability of both occurring is

$$P(A \text{ and } B) = P(A) \cdot P(B|A)$$

Example 28

At a university in western Pennsylvania, there were 5 burglaries reported in 2003, 16 in 2004, and 32 in 2005. If a researcher wishes to select at random two burglaries to further investigate, find the probability that both will have occurred in 2004

Solution

In this case, the events are dependent since the researcher wishes to investigate two distinct cases. Hence the first case is selected and not replaced.

P(C1 And C2)=P(C1)*P(C2\C1)= $\frac{16}{53} * \frac{15}{52} = \frac{60}{689}$

Example 29

he World Wide Insurance Company found that 53% of the residents of a city had homeowner's insurance (H) with the company. Of these clients, 27% also had automobile insurance (A) with the company. If a resident is selected at random, find the probability that the resident has both homeowner's and automobile insurance with the World Wide Insurance Company.

$$P(\text{H and A}) = P(\text{H}) \cdot P(\text{A}|\text{H}) = (0.53)(0.27) = 0.1431$$

Example 30

Three cards are drawn from an ordinary deck and not replaced. Find the probability of these events.

a. Getting 3 jacks

b. Getting an ace, a king, and a queen in order

c. Getting a club, a spade, and a heart in order

d. Getting 3 clubs

Solution

a. $P(\text{3 jacks}) = \frac{4}{52} \cdot \frac{3}{51} \cdot \frac{2}{50} = \frac{24}{132{,}600} = \frac{1}{5525}$

b. $P(\text{ace and king and queen}) = \frac{4}{52} \cdot \frac{4}{51} \cdot \frac{4}{50} = \frac{64}{132{,}600} = \frac{8}{16{,}575}$

c. $P(\text{club and spade and heart}) = \frac{13}{52} \cdot \frac{13}{51} \cdot \frac{13}{50} = \frac{2197}{132{,}600} = \frac{169}{10{,}200}$

d. $P(\text{3 clubs}) = \frac{13}{52} \cdot \frac{12}{51} \cdot \frac{11}{50} = \frac{1716}{132{,}600} = \frac{11}{850}$

6. Conditional Probability

The conditional probability of an event B *in relationship to an event* A *was defined as the probability that event* B *occurs after event* A *has already occurred. The conditional.*

probability of an event can be found by dividing both sides of the equation for multiplication rule 2 by P(A)*, as shown:*

$$P(A \text{ and } B) = P(A) \cdot P(B|A)$$

$$\frac{P(A \text{ and } B)}{P(A)} = \frac{\cancel{P(A)} \cdot P(B|A)}{\cancel{P(A)}}$$

$$\frac{P(A \text{ and } B)}{P(A)} = P(B|A)$$

Formula for Conditional Probability

The probability that the second event B occurs given that the first event A has occurred can be found by dividing the probability that both events occurred by the probability that the first event has occurred. The formula is

$$P(B|A) = \frac{P(A \text{ and } B)}{P(A)}$$

Example 31

A box contains black chips and white chips. A person selects two chips without replacement. If the probability of selecting a black chip *and* a white chip is $\frac{15}{56}$, and the probability of selecting a black chip on the first draw is $\frac{3}{8}$, find the probability of selecting the white chip on the second draw, *given* that the first chip selected was a black chip.

Solution

Let

B = selecting a black chip $\qquad$ W = selecting a white chip

Then

$$P(W|B) = \frac{P(B \text{ and } W)}{P(B)} = \frac{\frac{15}{56}}{\frac{3}{8}}$$

$$= \frac{15}{56} \div \frac{3}{8} = \frac{15}{56} \cdot \frac{8}{3} = \frac{\overset{5}{\cancel{15}}}{\underset{7}{\cancel{56}}} \cdot \frac{\overset{1}{\cancel{8}}}{\underset{1}{\cancel{3}}} = \frac{5}{7}$$

Hence, the probability of selecting a white chip on the second draw given that the first chip selected was black is $\frac{5}{7}$.

Example 32

The probability that Sam parks in a no-parking zone *and* gets a parking ticket is 0.06, and the probability that Sam cannot find a legal parking space and has to park in the no parking zone is 0.20. On Tuesday, Sam arrives at school and has to park in a no-parking zone. Find the probability that he will get a parking ticket.

Solution

Let

N = parking in a no-parking zone $\qquad$ T = getting a ticket

Then

$$P(T|N) = \frac{P(N \text{ and } T)}{P(N)} = \frac{0.06}{0.20} = 0.30$$

Hence, Sam has a 0.30 probability of getting a parking ticket, given that he parked in a no-parking zone.

7.COUNTING TECHNIQUES

Many times one wishes to know the number of all possible outcomes for a sequence of events. To determine this number, three rules can be used: the *fundamental counting rule*, the *permutation rule*, and the *combination rule*. These rules are explained here, The first rule is called the **fundamental counting rule.**

The Fundamental Counting Rule

Fundamental Counting Rule

In a sequence of n events in which the first one has k_1 possibilities and the second event has k_2 and the third has k_3, and so forth, the total number of possibilities of the sequence will be

$$k_1 \cdot k_2 \cdot k_3 \cdots k_n$$

Note: In this case *and* means to multiply.

A paint manufacturer wishes to manufacture several different paints. The categories include

Color: red, blue, white, black, green, brown yellow

Type: latex, oil

Texture: flat, semi-gloss, high gloss

Use: outdoor, indoor

How many different kinds of paint can be made if you can select one color, one type, one texture, and one use?

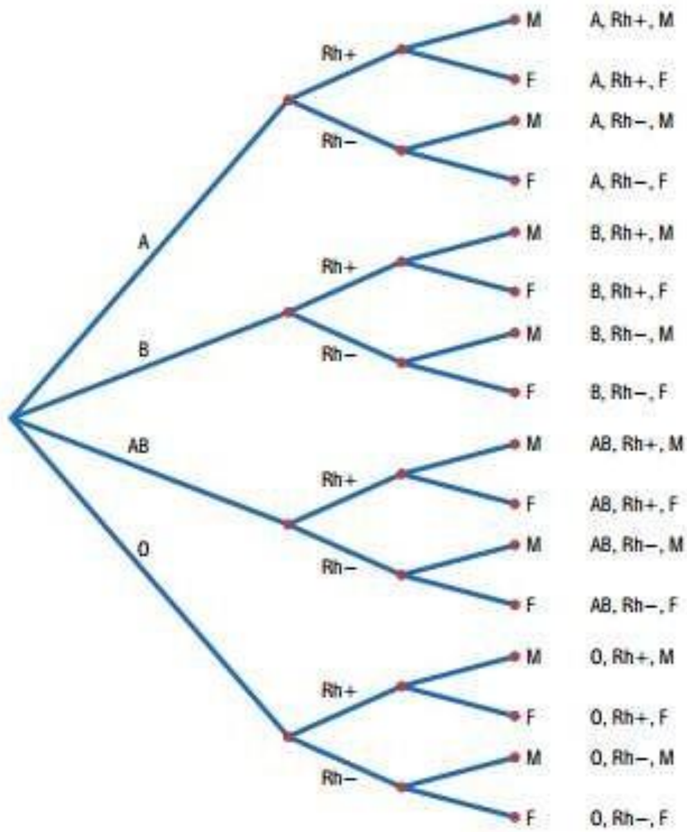
$$\left(\begin{array}{c}\#\text{ of}\\ \text{colors}\end{array}\right)\left(\begin{array}{c}\#\text{ of}\\ \text{types}\end{array}\right)\left(\begin{array}{c}\#\text{ of}\\ \text{textures}\end{array}\right)\left(\begin{array}{c}\#\text{ of}\\ \text{uses}\end{array}\right)$$

$$7 \cdot 2 \cdot 3 \cdot 2$$

84 different kinds of paint

Example33: Distribution of Blood Types

There are four blood types, A, B, AB, and O. Blood can also be Rh+ and Rh-. Finally, a blood donor can be classified as either male or female. How many different ways can a donor have his or her blood labeled?



Solution

Since there are 4 possibilities for blood type, 2 possibilities for Rh factor, and 2 possibilities for the gender of the donor, there are $4 \cdot 2 \cdot 2$, or 16, different classification categories, as shown.

Blood type		Rh		Gender	
4	·	2	·	2	= 16

Factorial *is the product of all the positive numbers from 1 to a number*

Factorial Formulas

For any counting n

$$n! = n(n-1)(n-2)\cdots 1$$
$$0! = 1$$

Permutations

A permutation is an arrangement of ***n*** objects in a specific order.

Combination

A selection of distinct objects without regard to order is called a

Permutation Rule

The arrangement of n objects in a specific order using r objects at a time is called a *permutation of n objects taking r objects at a time*. It is written as ${}_nP_r$, and the formula is

$${}_nP_r = \frac{n!}{(n-r)!}$$

Combination Rule

The number of combinations of r objects selected from n objects is denoted by ${}_nC_r$ and is given by the formula

$${}_nC_r = \frac{n!}{(n-r)!r!}$$

Example34:

Suppose a business owner has a choice of 5 locations in which to establish her business. She decides to rank each location according to certain criteria, such as price of the store and parking facilities. How many different ways can she rank the 5 locations?

Solution

There are

$$\binom{\text{first}}{\text{choice}}\binom{\text{second}}{\text{choice}}\binom{\text{third}}{\text{choice}}\binom{\text{fourth}}{\text{choice}}\binom{\text{fifth}}{\text{choice}}$$

$$5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$$

$5! = 5.\ 4.\ 3.\ 2.\ 1 = 120$ 120 different ways to rank the locations

different possible rankings. The reason is that she has 5 choices for the first location, 4 choices for the second location, 3 choices for the third location, etc.

Example 35

Given the letters A, B, C, and D, list the permutations and combinations for selecting two letters.

The permutations are

AB	BA	CA	DA
AC	BC	CB	DB
AD	BD	CD	DC

In permutations, AB is different from BA. But in combinations, AB is the same as BA since the order of the objects does not matter in combinations. Therefore, if duplicates are removed from a list of permutations, what is left is a list of combinations, as shown.

AB	~~BA~~	~~CA~~	~~DA~~
AC	BC	~~CB~~	~~DB~~
AD	BD	CD	~~DC~~

Hence the combinations of A, B, C, and D are AB, AC, AD, BC, BD, and CD. (Alternatively, BA could be listed and AB crossed out, etc.) The combinations have been listed alphabetically for convenience, but this is not a requirement.

Example 36

The advertising director for a television show has 7 ads to use on the program. If she selects 1 of them for the opening of the show, 1 for the middle of the show, and 1 for the ending of the show, how many possible ways can this be accomplished?

Solution

Since order is important, the solution is

$$_7P_3 = \frac{7!}{(7-3)!} = \frac{7!}{4!} = 210$$

Hence, there would be 210 ways to show 3 ads.

Example 37:

A school musical director can select 2 musical plays to present next year. One will be presented in the fall, and one will be presented in the spring. If she has 9 to pick from, how many different possibilities are there?

Solution

Order is important since one play can be presented in the fall and the other play in the spring.

$$ {}_9P_2 = \frac{9!}{(9-2)!} = \frac{9!}{7!} = \frac{9 \cdot 8 \cdot 7!}{7!} = 72 $$

There are 72 different possibilities

Example 38

How many combinations of 4 objects are there, taken 2 at a time?

Solution

Since this is a combination problem, the answer is

$$ {}_4C_2 = \frac{4!}{(4-2)!2!} = \frac{4!}{2!2!} = \frac{\overset{2}{\cancel{4}} \cdot 3 \cdot \cancel{2!}}{\cancel{2} \cdot 1 \cdot \cancel{2!}} = 6 $$

Example 39

A newspaper editor has received 8 books to review. He decides that he can use 3 reviews in his newspaper. How many different ways can these 3 reviews be selected?

Solution

$$ {}_8C_3 = \frac{8!}{(8-3)!3!} = \frac{8!}{5!3!} = \frac{8 \cdot 7 \cdot 6}{3 \cdot 2 \cdot 1} = 56 $$

There are 56 possibilities

Example 40

In a club there are 7 women and 5 men. A committee of 3 women and 2 men is to be chosen. How many different possibilities are there?

Solution

Here, one must select 3 women from 7 women, which can be done in 7C3, or 35, ways. Next, 2 men must be selected from 5 men, which can be done in 5C2, or 10, ways. Finally, by the fundamental counting rule, the total number of different ways is 35 ***?*** 10 ***5*** 350, since one is choosing both men and women. Using the formula gives

$$_7C_3 \cdot {_5C_2} = \frac{7!}{(7-3)!3!} \cdot \frac{5!}{(5-2)!2!} = 350$$

8. Bayes theorem

- Bayes' Theorem is a way of finding a probability when we know certain other probabilities.
- If we have two events A and B, and we are given the conditional probability of A given B, denoted P(A|B), we can use Bayes' Theorem to find P(B|A), the conditional probability of B given A.

$$P(A|B)=\frac{P(A)*P(B|A)}{P(B)}$$

Which tells us:	how often A happens *given that B happens*, written **P(A|B)**,
When we know:	how often B happens *given that A happens*, written **P(B|A)**
	and how likely A is on its own, written **P(A)**
	and how likely B is on its own, written **P(B)**

Note

Let us say P(Fire) means how often there is fire, and P(Smoke) means how often we see smoke, then:

P(Fire| Smoke) means how often there is fire when we can see smoke
P(Smoke| Fire) means how often we can see smoke when there is fire

So the formula kind of tells us "forwards" P(Fire| Smoke) when we know "backwards" P(Smoke| Fire)

Example 41

- dangerous fires are rare (1%)
- but smoke is fairly common (10%) due to barbecues,
- and 90% of dangerous fires make smoke

We can then discover the probability of dangerous Fire when there is Smoke

$$P(Fire|Smoke) = \frac{P(Fire)\,P(Smoke|Fire)}{P(Smoke)}$$
$$= \frac{1\% \times 90\%}{10\%}$$
$$= 9\%$$

Example 42

You are planning a picnic today, but the morning is cloudy

- Oh no! 50% of all rainy days start off cloudy!
- But cloudy mornings are common (about 40% of days start cloudy)
- And this is usually a dry month (only 3 of 30 days tend to be rainy, or 10%)

What is the chance of rain during the day?

We will use Rain to mean rain during the day, and Cloud to mean cloudy morning.

The chance of Rain given Cloud is written P(Rain|Cloud)

So let's put that in the formula:

$$P(Rain|\ Cloud) = P(Rain)\ P(Cloud\ |Rain) /\ P(Cloud)$$

- P(Rain) is Probability of Rain = 10%
- P(Cloud| Rain) is Probability of Cloud, given that Rain happens = 50%
- P(Cloud) is Probability of Cloud = 40%

$$P(Rain|\ Cloud) = \frac{0.1\ x\ 0.5}{0.4} = .125$$

9. Discrete Probability Distributions

- Random Variable *A* random variable *is a variable whose value is determined by the outcome of a random experiment.*

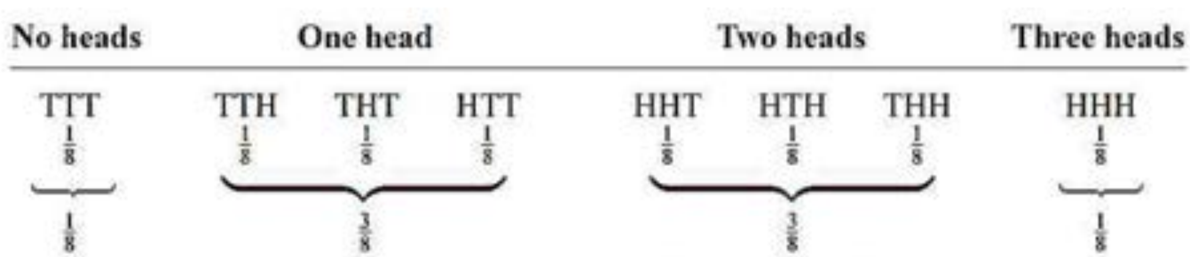
A ***random variable*** is a variable whose values are determined by chance

- **A *discrete probability* distribution consists of the values a random variable can assume and the corresponding probabilities of the values.**
- **The sum of the probabilities of all events in a sample space add up to 1. Each probability is between 0 and 1, inclusively.**

Example 1

Recall that when three coins are tossed, the sample space is represented as TTT, TTH, THT, HTT, HHT, HTH, THH, HHH; and if *X* is the random variable for the number of heads, then *X* assumes the value 0, 1, 2, or 3.

Probabilities for the values of *X* can be determined as follows:



Hence, the probability of getting no heads is $\frac{1}{8}$, one head is $\frac{3}{8}$, two heads is $\frac{3}{8}$, and three heads is $\frac{1}{8}$. From these values, a probability distribution can be constructed by listing the outcomes and assigning the probability of each outcome, as shown here.

Number of heads X	0	1	2	3
Probability $P(X)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

Example 2- *Rolling a Die*

Construct a probability distribution for rolling a single die.

Solution

Since the sample space is 1, 2, 3, 4, 5, 6 and each outcome has a probability of, the distribution is as shown.

Outcome X	1	2	3	4	5	6
Probability $P(X)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

Probability distributions can be shown graphically by representing the values of X on the x axis and the probabilities $P(X)$ on the y axis.

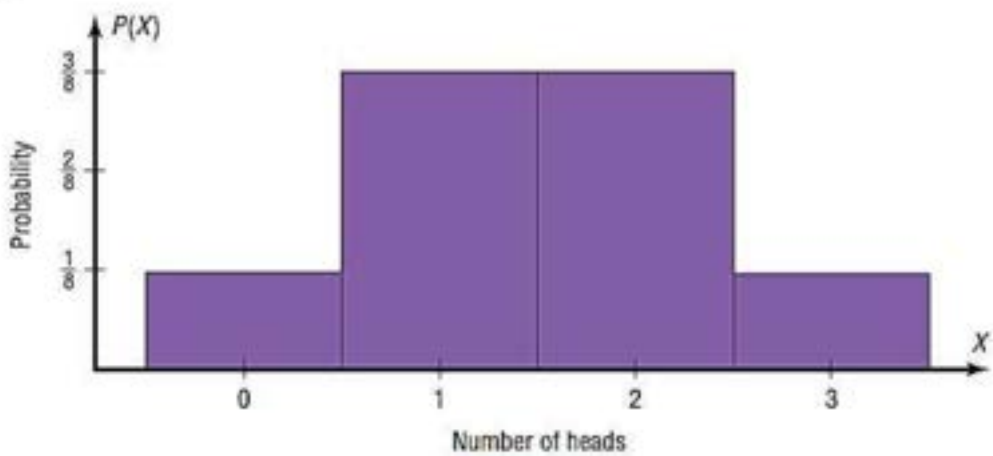
Example 3- Tossing Coins

Represent graphically the probability distribution for the sample space for tossing three coins.

Number of heads X	0	1	2	3
Probability $P(X)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

Solution

The values that X assumes are located on the x axis, and the values for $P(X)$ are located on the y axis.



Two Requirements for a Probability Distribution

1. The sum of the probabilities of all the events in the sample space must equal 1; that is, $\Sigma P(X) = 1$.
2. The probability of each event in the sample space must be between or equal to 0 and 1. That is, $0 \le P(X) \le 1$.

Example 4- Probability Distributions

Determine whether each distribution is a probability distribution.

a.

X	0	5	10	15	20
$P(X)$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$

b.

X	0	2	4	6
$P(X)$	−1.0	1.5	0.3	0.2

c.

X	1	2	3	4
$P(X)$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{9}{16}$

d.

X	2	3	7
$P(X)$	0.5	0.3	0.4

Solution

a. *Yes, it is a probability distribution.*

b. *No, it is not a probability distribution, since P(X) cannot be 1.5 or _1.0.*

c. *Yes, it is a probability distribution.*

d. *No, it is not, since ∑ □(□) = 1.2.*

Mean, Variance, Standard Deviation, and Expectation

The mean, variance, and standard deviation for a probability distribution are computed differently from the mean, variance, and standard deviation for samples.

Mean

the mean for a sample or population was computed by adding the values and dividing by the total number of values, as shown in these formulas:

$$X = \frac{\sum X}{n} \quad , \mu = \frac{\sum X}{N}$$

Formula for the Mean of a Probability Distribution

The mean of a random variable with a discrete probability distribution is

$$\mu = X_1 \cdot P(X_1) + X_2 \cdot P(X_2) + X_3 \cdot P(X_3) + \cdots + X_n \cdot P(X_n)$$
$$= \Sigma X \cdot P(X)$$

where $X_1, X_2, X_3, \ldots, X_n$ are the outcomes and $P(X_1), P(X_2), P(X_3), \ldots, P(X_n)$ are the corresponding probabilities.

Note: $\Sigma X \cdot P(X)$ means to sum the products.

Example 5–Rolling a Die

Find the mean of the number of spots that appear when a die is tossed.

Solution

In the toss of a die, the mean can be computed thus.

Outcome X	1	2	3	4	5	6
Probability $P(X)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

$$\mu = \sum X \cdot P(X)$$
$$= 1 \cdot \tfrac{1}{6} + 2 \cdot \tfrac{1}{6} + 3 \cdot \tfrac{1}{6} + 4 \cdot \tfrac{1}{6} + 5 \cdot \tfrac{1}{6} + 6 \cdot \tfrac{1}{6}$$
$$= \tfrac{21}{6} = 3\tfrac{1}{2} \text{ or } 3.5$$

Example 6– Children in a Family

In a family with two children, find the mean of the number of children who will be girls.

Solution

The probability distribution is as follows:

Number of girls X	0	1	2
Probability P(X)	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

Hence, the mean is

$$\mu = \Sigma X \cdot P(X) = 0 \cdot \tfrac{1}{4} + 1 \cdot \tfrac{1}{2} + 2 \cdot \tfrac{1}{4} = 1$$

Example 7– Tossing Coins

If three coins are tossed, find the mean of the number of heads that occur.

Solution

The probability distribution is

Number of heads X	0	1	2	3
Probability P(X)	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

The mean is

$$\mu = \Sigma X \cdot P(X) = 0 \cdot \tfrac{1}{8} + 1 \cdot \tfrac{3}{8} + 2 \cdot \tfrac{3}{8} + 3 \cdot \tfrac{1}{8} = \tfrac{12}{8} = 1\tfrac{1}{2} \text{ or } 1.5$$

The value 1.5 cannot occur as an outcome. Nevertheless, it is the long-run or theoretical average.

Example 8– Number of Trips of Five Nights or More

The probability distribution shown represents the number of trips of five nights or more that American adults take per year. (That is, 6% do not take any trips lasting five nights or more, 70% take one trip lasting five nights or more per year, etc.) Find the mean.

Number of trips X	0	1	2	3	4
Probability P(X)	0.06	0.70	0.20	0.03	0.01

Solution

$$\begin{aligned}\mu &= \Sigma X \cdot P(X)\\ &= (0)(0.06) + (1)(0.70) + (2)(0.20) + (3)(0.03) + (4)(0.01)\\ &= 0 + 0.70 + 0.40 + 0.09 + 0.04\\ &= 1.23 \approx 1.2\end{aligned}$$

Hence, the mean of the number of trips lasting five nights or more per year taken by American adults is 1.2

Variance and Standard Deviation

To find the variance for the random variable of a probability distribution, subtract the theoretical mean of the random variable from each outcome and square the difference. Then multiply each difference by its corresponding probability and add the products. The formula is

$$\sigma^2 = \Sigma[(X - \mu)^2 \cdot P(X)]$$

Formula for the Variance of a Probability Distribution

Find the variance of a probability distribution by multiplying the square of each outcome by its corresponding probability, summing those products, and subtracting the square of the mean. The formula for the variance of a probability distribution is

$$\sigma^2 = \Sigma[X^2 \cdot P(X)] - \mu^2$$

The standard deviation of a probability distribution is

$$\sigma = \sqrt{\sigma^2} \quad \text{or} \quad \sqrt{\Sigma[X^2 \cdot P(X)] - \mu^2}$$

Example 9–Rolling a Die

Compute the variance and standard deviation for the probability distribution in Example 4.

Solution

Recall that the mean is μ = 3.5, *as computed in Example 4. Square each outcome and multiply by the corresponding probability, sum those products, and then subtract the square of the mean.*

$$\sigma^2 = (1^2 \cdot \tfrac{1}{6} + 2^2 \cdot \tfrac{1}{6} + 3^2 \cdot \tfrac{1}{6} + 4^2 \cdot \tfrac{1}{6} + 5^2 \cdot \tfrac{1}{6} + 6^2 \cdot \tfrac{1}{6}) - (3.5)^2 = 2.9$$

To get the standard deviation, find the square root of the variance.

$$\sigma = \sqrt{2.9} = 1.7$$

Example 10–Selecting Numbered Balls

A box contains 5 balls. Two are numbered 3, one is numbered 4, and two are numbered 5. The balls are mixed and one is selected at random. After a ball is selected, its number is recorded. Then it is replaced. If the experiment is repeated many times, find the variance and standard deviation of the numbers on the balls.

Solution

Let X be the number on each ball. The probability distribution is

Number on ball X	3	4	5
Probability $P(X)$	$\frac{2}{5}$	$\frac{1}{5}$	$\frac{2}{5}$

The mean is

$$\mu = \Sigma X \cdot P(X) = 3 \cdot \tfrac{2}{5} + 4 \cdot \tfrac{1}{5} + 5 \cdot \tfrac{2}{5} = 4$$

The variance is

$$\begin{aligned} \sigma &= \Sigma[X^2 \cdot P(X)] - \mu^2 \\ &= 3^2 \cdot \tfrac{2}{5} + 4^2 \cdot \tfrac{1}{5} + 5^2 \cdot \tfrac{2}{5} = 4 \\ &= 16\tfrac{4}{5} - 16 \\ &= \tfrac{4}{5} \end{aligned}$$

The standard deviation is

$$\sigma = \sqrt{\frac{4}{5}} = \sqrt{0.8} = 0.894$$

The mean, variance, and standard deviation can also be found by using vertical columns, as shown.

X	$P(X)$	$X \cdot P(X)$	$X^2 \cdot P(X)$
3	0.4	1.2	3.6
4	0.2	0.8	3.2
5	0.4	2.0	10
		$\Sigma X \cdot P(X) = 4.0$	16.8

$\sigma^2 = 16.8 - 4^2 = 16.8 - 16 = 0.8$

and

$\sigma = \sqrt{0.8} = 0.894$

Example 11- On Hold for Talk Radio

A talk radio station has four telephone lines. If the host is unable to talk (i.e., during a commercial) or is talking to a person, the other callers are placed on hold. When all lines are in use, others who are trying to call in get a busy signal. The probability that 0, 1, 2, 3, or 4

people will get through is shown in the distribution. Find the variance and standard deviation for the distribution.

X	0	1	2	3	4
$P(X)$	0.18	0.34	0.23	0.21	0.04

Should the station have considered getting more phone lines installed?

Solution

The mean is

$$\mu = \Sigma X \cdot P(X)$$
$$= 0 \cdot (0.18) + 1 \cdot (0.34) + 2 \cdot (0.23) + 3 \cdot (0.21) + 4 \cdot (0.04)$$
$$= 1.6$$

The variance is

$$\sigma^2 = \Sigma[X^2 \cdot P(X)] - \mu^2$$
$$= [0^2 \cdot (0.18) + 1^2 \cdot (0.34) + 2^2 \cdot (0.23) + 3^2 \cdot (0.21) + 4^2 \cdot (0.04)] - 1.6^2$$
$$= [0 + 0.34 + 0.92 + 1.89 + 0.64] - 2.56$$
$$= 3.79 - 2.56 = 1.23$$
$$= 1.2 \text{ (rounded)}$$

The standard deviation is $\sigma = \sqrt{\sigma^2}$, or $\sigma = \sqrt{1.2} = 1.1$.

Expectation

Another concept related to the mean for a probability distribution is that of expected value or expectation. Expected value is used in various types of games of chance, in insurance, and in other areas, such as decision theory.

The **expected value** of a discrete random variable of a probability distribution is the theoretical average of the variable. The formula is

$$\mu = E(X) = \Sigma X \cdot P(X)$$

The symbol $E(X)$ is used for the expected value.

The formula for the expected value is the same as the formula for the theoretical mean. The expected value, then, is the theoretical mean of the probability distribution. That is, $E(X) = \mu$.

Example 12- *Selecting Balls*

You have six balls numbered 1-8 and 13 are placed in a box. A ball is selected at random, and its number is recorded and it is replaced Find the expected value of the number that will occur

Number (X)	1	2	3	5	8	13
Probability $P(X)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

$$E(X) = 1 \cdot \tfrac{1}{6} + 2 \cdot \tfrac{1}{6} + 3 \cdot \tfrac{1}{6} + 5 \cdot \tfrac{1}{6} + 8 \cdot \tfrac{1}{6} + 13 \cdot \tfrac{1}{6} = 5\tfrac{1}{3}$$

10. Binomial Distribution

Many types of probability problems have only two outcomes or can be reduced to two outcomes. For example, when a coin is tossed, it can land heads or tails. When a baby is born, it will be either male or female. In a basketball game, a team either wins or loses. A true/false item can be answered in only two ways, true or false.

A *binomial experiment* is a probability experiment that satisfies the following four requirements:

1. There must be a fixed number of trials.

2. Each trial can have only two outcomes or outcomes that can be reduced to two

outcomes. These outcomes can be considered as either success or failure.

3. The outcomes of each trial must be independent of one another.

4. The probability of a success must remain the same for each trial.

A binomial experiment and its results give rise to a special probability distribution called the ***binomial distribution*****.**

The outcomes of a binomial experiment and the corresponding probabilities of these outcomes are called a *binomial distribution*.

Notation for the Binomial Distribution

$P(S)$	The symbol for the probability of success
$P(F)$	The symbol for the probability of failure
p	The numerical probability of a success
q	The numerical probability of a failure

$$P(S) = p \quad \text{and} \quad P(F) = 1 - p = q$$

n	The number of trials
X	The number of successes in n trials

Note that $0 \le X \le n$ and $X = 0, 1, 2, 3, \ldots, n$.

The probability of a success in a binomial experiment can be computed with this formula.

Binomial Probability Formula

In a binomial experiment, the probability of exactly X successes in n trials is

$$P(X) = \frac{n!}{(n-X)!X!} \cdot p^X \cdot q^{n-X}$$

Example 13- Survey on Doctor Visits

A survey found that one out of five Americans say he or she has visited a doctor in any given month. If 10 people are selected at random, find the probability that exactly 3 will have visited a doctor last month.

Solution

In this case, $n = 10$, $X = 3$, $p = \frac{1}{5}$, and $q = \frac{4}{5}$. Hence,

$$P(3) = \frac{10!}{(10-3)!3!}\left(\frac{1}{5}\right)^3\left(\frac{4}{5}\right)^7 = 0.201$$

Example 14- Survey on Employment

A survey from Teenage Research Unlimited (Northbrook, Illinois) found that 30% of teenage consumers receive their spending money from part-time jobs. If 5 teenagers are selected at random, find the probability that at least 3 of them will have part-time jobs.

Solution

To find the probability that at least 3 have part-time jobs, it is necessary to find the individual probabilities for 3, or 4, or 5, and then add them to get the total probability.

$$P(3) = \frac{5!}{(5-3)!3!}(0.3)^3(0.7)^2 = 0.132$$

$$P(4) = \frac{5!}{(5-4)!4!}(0.3)^4(0.7)^1 = 0.028$$

$$P(5) = \frac{5!}{(5-5)!5!}(0.3)^5(0.7)^0 = 0.002$$

Hence,

P(at least three teenagers have part-time jobs)
$= 0.132 + 0.028 + 0.002 = 0.162$

Mean, Variance, and Standard Deviation for the Binomial Distribution

The mean, variance, and standard deviation of a variable that has the *binomial distribution* can be found by using the following formulas.

Mean: $\mu = n \cdot p$ Variance: $\sigma^2 = n \cdot p \cdot q$ Standard deviation: $\sigma = \sqrt{n \cdot p \cdot q}$

Example 15- Tossing a Coin

A coin is tossed 4 times. Find the mean, variance, and standard deviation of the number of heads that will be obtained.

Solution

With the formulas for the binomial distribution and $n = 4$, $p = \frac{1}{2}$, and $q = \frac{1}{2}$, the results are

$$\mu = n \cdot p = 4 \cdot \tfrac{1}{2} = 2$$
$$\sigma^2 = n \cdot p \cdot q = 4 \cdot \tfrac{1}{2} \cdot \tfrac{1}{2} = 1$$
$$\sigma = \sqrt{1} = 1$$

From Example 15, when four coins are tossed many, many times, the average of the number of heads that appear is 2, and the standard

deviation of the number of heads is 1. Note that these are theoretical values. As stated previously, this problem can be solved by using the formulas for expected value. The distribution is shown.

No. of heads X	0	1	2	3	4
Probability $P(X)$	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$	$\frac{1}{16}$

$$\mu = E(X) = \Sigma X \cdot P(X) = 0 \cdot \tfrac{1}{16} + 1 \cdot \tfrac{4}{16} + 2 \cdot \tfrac{6}{16} + 3 \cdot \tfrac{4}{16} + 4 \cdot \tfrac{1}{16} = \tfrac{32}{16} = 2$$

$$\sigma^2 = \Sigma X^2 \cdot P(X) - \mu^2$$

$$= 0^2 \cdot \tfrac{1}{16} + 1^2 \cdot \tfrac{4}{16} + 2^2 \cdot \tfrac{6}{16} + 3^2 \cdot \tfrac{4}{16} + 4^2 \cdot \tfrac{1}{16} - 2^2 = \tfrac{80}{16} - 4 = 1$$

$$\sigma = \sqrt{1} = 1$$

Hence, the simplified binomial formulas give the same results.

Example 16- Rolling a Die

A die is rolled 360 times. Find the mean, variance, and standard deviation of the number of 4s that will be rolled.

Solution

This is a binomial experiment since getting a 4 is a success and not getting a 4 is considered a failure. Hence $n = 360$, $p = \frac{1}{6}$, and $q = \frac{5}{6}$.

$$\mu = n \cdot p = 360 \cdot \tfrac{1}{6} = 60$$

$$\sigma^2 = n \cdot p \cdot q = 360 \cdot \left(\tfrac{1}{6}\right)\left(\tfrac{5}{6}\right) = 50$$

$$\sigma = \sqrt{n \cdot p \cdot q} = \sqrt{50} = 7.07$$

Example 17- Likelihood of Twins

The Statistical Bulletin *published by Metropolitan Life Insurance Co. reported that 2% of all-American births result in twins. If a random* *sample of 8000 births is taken, find the mean, variance, and standard deviation of the number of births that would result in twins.*

Solution

This is a binomial situation, since a birth can result in either twins or not twins (i.e., two outcomes).

$$\mu = n \cdot p = (8000)(0.02) = 160$$
$$\sigma^2 = n \cdot p \cdot q = (8000)(0.02)(0.98) = 156.8$$
$$\sigma = \sqrt{n \cdot p \cdot q} = \sqrt{156.8} = 12.5$$

For the sample, the average number of births that would result in twins is 160, the variance is 156.8, or 157, and the standard deviation is 12.5, or 13 if rounded.

11. The Multinomial Distribution

- ■ *The* ***multinomial distribution*** *is similar to the binomial distribution but has the advantage of allowing one to compute probabilities when there are more than two outcomes. For example, a survey might require the responses of "approve," "disapprove," or "no opinion."*
- ■ *The binomial distribution is a special case of the multinomial distribution.*

Formula for the Multinomial Distribution

If X consists of events $E_1, E_2, E_3, \ldots, E_k$, which have corresponding probabilities $p_1, p_2, p_3, \ldots, p_k$ of occurring, and X_1 is the number of times E_1 will occur, X_2 is the number of times E_2 will occur, X_3 is the number of times E_3 will occur, etc., then the probability that X will occur is

$$P(X) = \frac{n!}{X_1! \cdot X_2! \cdot X_3! \cdots X_k!} \cdot p_1^{X_1} \cdot p_2^{X_2} \cdots p_k^{X_k}$$

where $X_1 + X_2 + X_3 + \cdots + X_k = n$ and $p_1 + p_2 + p_3 + \cdots + p_k = 1$.

Example 18–Leisure Activities

In a large city, 50% of the people choose a movie, 30% choose dinner and a play, and 20% choose shopping as a leisure activity. If a sample of 5 people is randomly selected, find the probability that 3 are planning to go to a movie, 1 to a play, and 1 to a shopping mall.

Solution

We know that $n = 5$, $X_1 = 3$, $X_2 = 1$, $X_3 = 1$, $p_1 = 0.50$, $p_2 = 0.30$, and $p_3 = 0.20$. Substituting in the formula gives

$$P(X) = \frac{5!}{3! \cdot 1! \cdot 1!} \cdot (0.50)^3(0.30)^1(0.20)^1 = 0.15$$

Example 19–CD Purchases

In a music store, a manager found that the probabilities that a person buys 0, 1, or 2 or more CDs are 0.3, 0.6, and 0.1, respectively. If 6 customers enter the store, find the probability that 1 won’t buy any CDs, 3 will buy 1 CD, and 2 will buy 2 or more CDs.

Solution

It is given that $n = 6, X_1 = 1, X_2 = 3, X_3 = 2, p_1 = 0.3, p_2 = 0.6$, and $p_3 = 0.1$. Then

$$P(X) = \frac{6!}{1!3!2!} \cdot (0.3)^1(0.6)^3(0.1)^2$$
$$= 60 \cdot (0.3)(0.216)(0.01) = 0.03888$$

Example 20–Selecting Colored Balls

A box contains 4 white balls, 3 red balls, and 3 blue balls. A ball is selected at random, and its color is written down. It is replaced each time. Find the probability that if 5 balls are selected, 2 are white, 2 are red, and 1 is blue

We know that $n = 5, X_1 = 2, X_2 = 2, X_3 = 1; p_1 = \frac{4}{10}, p_2 = \frac{3}{10}$, and $p_3 = \frac{3}{10}$; hence,

$$P(X) = \frac{5!}{2!2!1!} \cdot \left(\frac{4}{10}\right)^2\left(\frac{3}{10}\right)^2\left(\frac{3}{10}\right)^1 = \frac{81}{625}$$

12. The Poisson Distribution

A discrete probability distribution that is useful when n *is large and* p *is small and when the independent variables occur over a period of time is called the* ***Poisson distribution.***

Formula for the Poisson Distribution

The probability of X occurrences in an interval of time, volume, area, etc., for a variable where λ (Greek letter lambda) is the mean number of occurrences per unit (time, volume, area, etc.) is

$$P(X; \lambda) = \frac{e^{-\lambda}\lambda^X}{X!} \qquad \text{where } X = 0, 1, 2, \ldots$$

The letter e is a constant approximately equal to 2.7183.

Example 21–Typographical Errors

If there are 200 typographical errors randomly distributed in a 500-page manuscript, find the probability that a given page contains exactly 3 errors.

Solution

First, find the mean number λ of errors. Since there are 200 errors distributed over 500 pages, each page has an average of

$$\lambda = \frac{200}{500} = \frac{2}{5} = 0.4$$

or 0.4 error per page. Since $X = 3$, substituting into the formula yields

$$P(X; \lambda) = \frac{e^{-\lambda}\lambda^X}{X!} = \frac{(2.7183)^{-0.4}(0.4)^3}{3!} = 0.0072$$

Thus, there is less than a 1% chance that any given page will contain exactly 3 errors.

Example 22–Toll-Free Telephone Calls

A sales firm receives, on average, 3 calls per hour on its toll-free number. For any given hour, find the probability that it will receive the following.

a. *At most 3 calls* b. *At least 3 calls* c. *5 or more calls*

Solution

a. "At most 3 calls" means 0, 1, 2, or 3 calls. Hence,

$$P(0; 3) + P(1; 3) + P(2; 3) + P(3; 3)$$
$$= 0.0498 + 0.1494 + 0.2240 + 0.2240$$
$$= 0.6472$$

b. "At least 3 calls" means 3 or more calls. It is easier to find the probability of 0, 1, and 2 calls and then subtract this answer from 1 to get the probability of at least 3 calls.

$$P(0; 3) + P(1; 3) + P(2; 3) = 0.0498 + 0.1494 + 0.2240 = 0.4232$$

and

$$1 - 0.4232 = 0.5768$$

c. For the probability of 5 or more calls, it is easier to find the probability of getting 0, 1, 2, 3, or 4 calls and subtract this answer from 1. Hence,

$$P(0; 3) + P(1; 3) + P(2; 3) + P(3; 3) + P(4; 3)$$
$$= 0.0498 + 0.1494 + 0.2240 + 0.2240 + 0.1680$$
$$= 0.8152$$

and

$$1 - 0.8152 = 0.1848$$

The Poisson distribution can also be used to approximate the binomial distribution when the expected value $\lambda = n \cdot p$ is less than 5, as shown in Example 5–29. (The same is true when $n \cdot q < 5$.)

Example 23–Left-Handed People

If approximately 2% of the people in a room of 200 people are left-handed, find the probability that exactly 5 people there are left-handed.

Solution

Since $\lambda = n \cdot p$, then $\lambda = (200)(0.02) = 4$. Hence,

$$P(X; \lambda) = \frac{(2.7183)^{-4}(4)^5}{5!} = 0.1563$$

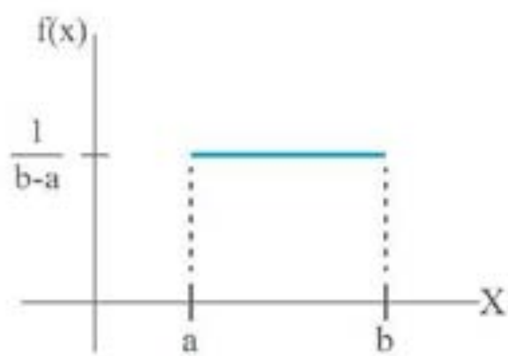
Continuous Probability Distributions

13. Uniform distribution

The uniform distribution is a continuous probability distribution and is concerned with events that are equally likely to occur. When working out problems that have a uniform distribution, be careful to note if the data is inclusive or exclusive.

$$f(x) = \frac{1}{b-a}$$

for two constants a and b, such that $a < x < b$. A graph of the p.d.f. looks like this:



The notation for the uniform distribution is

$X \sim U(a,b)$ where $a =$ the lowest value of x and $b =$ the highest value of x.

The probability density function is $f(x) = \frac{1}{b-a}$ for $a \le x \le b$.

For this example, $X \sim U(0,23)$ and $f(x) = \frac{1}{23-0}$ for $0 \le X \le 23$.

Formulas for the theoretical mean and standard deviation are

$$\mu = \frac{a+b}{2}$$

$$\sigma = \sqrt{\frac{(b-a)^2}{12}}$$

Example 24

Let a=0.b=14 where X~(0,14)

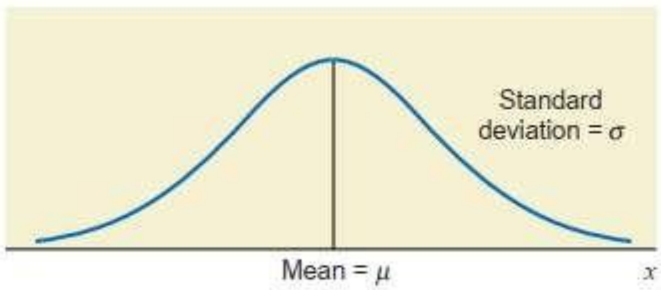
Find mean and variance

$\mu = (0+14)/2 = 7$, $\boldsymbol{\sigma} = \sqrt{\frac{(14-0)^2}{12}}$ =2,020

14. The Normal Distribution

The normal distribution is one of the many probability distributions that a continuous random variable can possess. The normal distribution is the most important and most widely used of all probability distributions. A large number of phenomena in the real world are normally distributed either exactly or approximately. The continuous random variables representing heights and weights of people, scores on an examination, weights of packages

The *normal probability distribution* or the normal curve is a bell-shaped (symmetric) curve. Such a curve is shown in Figure below. Its mean is denoted by μ and its standard deviation by σ. A continuous random variable x that has a normal distribution is called a normal random variable. Note that not all bell-shaped curves represent a normal distribution curve. Only a specific kind of bell-shaped curve represents a normal curve.

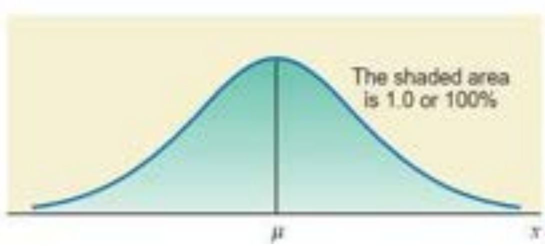


Normal Probability Distribution A *normal probability distribution*, when plotted, gives a bell-shaped curve such that:

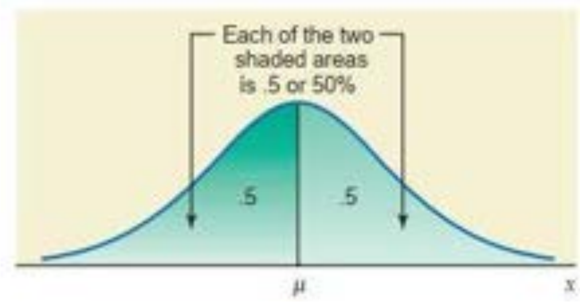
1. The total area under the curve is 1.0.
2. The curve is symmetric about the mean.
3. The two tails of the curve extend indefinitely.

A normal distribution possesses the following three characteristics:

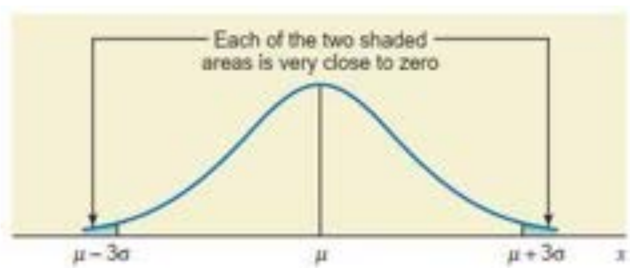
1. The total area under a normal distribution curve is 1.0, or 100%, as shown in Figure



2. *A normal distribution curve is symmetric about the mean, as shown in Figure Consequently, 50% of the total area under a normal distribution curve lies on the left side of the mean, and 50% lies on the right side of the mean*

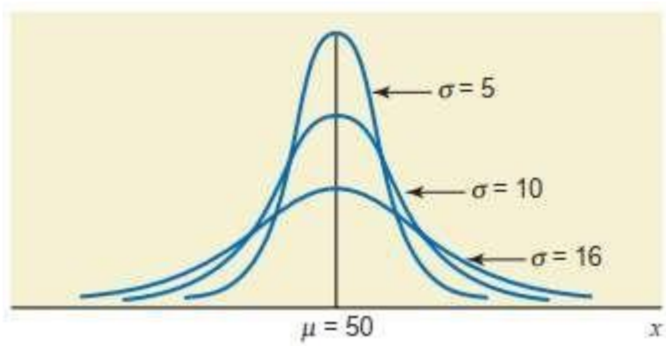


3. *The tails of a normal distribution curve extend indefinitely in both directions without touching or crossing the horizontal axis. Although a normal distribution curve never meets the horizontal axis, beyond the points represented by* m- *3* □ *and* m + 3□ *it becomes so close to this axis that the area under the curve beyond these points in both directions can be taken as virtually zero. These areas are shown in Figure*



The mean, μ, and the standard deviation, □, are the parameters of the normal distribution. Given the values of these two parameters, we can find the area under a normal distribution curve for any interval. Remember, there is not just one normal distribution curve but a family of normal distribution curves. Each different set of values of μ and □ gives a different normal distribution. The value of μ determines the center of a normal distribution curve on the horizontal axis, and the value of *s* gives the spread of the normal distribution curve. The three

normal distribution curves drawn in Figure below have the same mean but different standard deviations.



The equation of the normal distribution is

where *e* 5 2.71828 and *p* 5 3.14159 approximately; *f*(*x*), called the probability density function, gives the vertical distance between the horizontal axis and the curve at point *x*. For the information of those who are familiar with integral calculus, the definite integral of this equation from *a* to *b* gives the probability that *x* assumes a value between *a* and *b*.

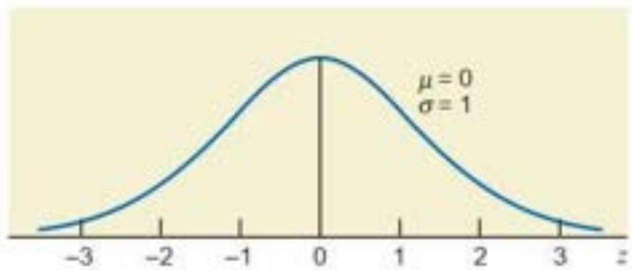
$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(1/2)[(x-\mu)/\sigma]^2}$$

The Standard Normal Distribution

The *standard normal distribution* is a special case of the normal distribution. For the standard normal distribution, the value of the mean is equal to zero, and the value of the standard deviation is equal to 1.

Figure below displays the standard normal distribution curve. The random variable that possesses the standard normal distribution is denoted by *z*. In

other words, the units for the standard normal distribution curve are denoted by *z* and are called the z *values* or z *scores*. They are also called ***standard units*** or ***standard scores***

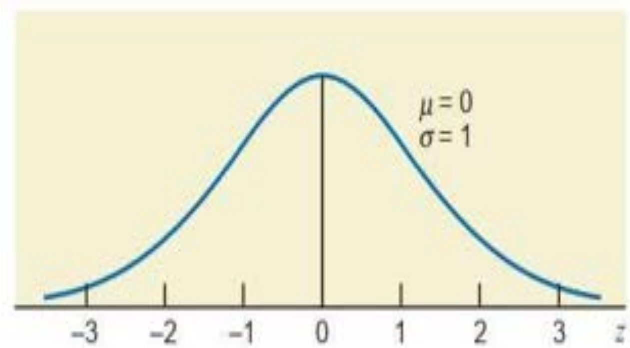


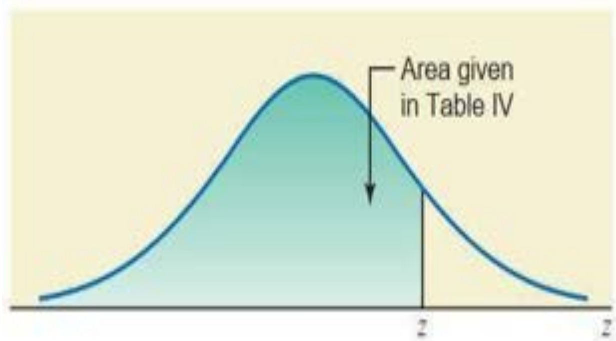
Definition

z Values or z Scores The units marked on the horizontal axis of the standard normal curve are denoted by *z* and are called the *z values* or *z scores*. A specific value of *z* gives the distance between the mean and the point represented by *z* in terms of the standard deviation.

In Figure below , the horizontal axis is labeled *z*. The *z* values on the right side of the mean are positive and those on the left side are negative. ***The z value for a point on the horizontal axis gives the distance between the mean and that point in terms of the standard deviation.*** For example, a point with a value of *z* 5 2 is two standard deviations to the right of the mean. Similarly, a point with a value of *z* 5 22 is two standard deviations to the left of the mean.

The standard normal distribution table, Table IV of Appendix C, lists the areas under the standard normal curve to the left of *z* values from to 3.49. To read the standard normal distribution table, we look for the given *z* value in the table and record the value corresponding to that *z* value. As shown in Figure below, Table IV gives what is called the cumulative probability to the left of any *z* value.





Although the values of z on the left side of the mean are negative, the area under the curve is always positive

Example 26

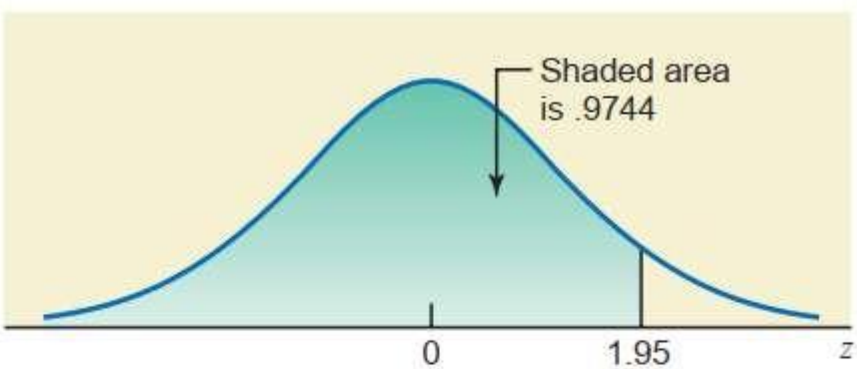
Find the area under the standard normal curve to the left of 1.95

Solution We divide the given number 1.95 into two portions: 1.9 (the digit before the decimal and one digit after the decimal) and .05 (the second digit after the decimal). (Note that $1.95 = 1.9 + .05$.) To find the required area under the standard normal curve, we locate 1.9 in the column for z on the left side of Table IV and .05 in the row for z at the top of Table IV. The entry where the row for 1.9 and the column for .05 intersect gives the area under the standard normal curve to the left of $z = 1.95$. The relevant portion of Table IV is reproduced as Table 6.2. From Table IV or Table 6.2, the entry where the row for 1.9 and the column for .05 cross is .9744. Consequently, the area under the standard normal curve to the left of $z = 1.95$ is .9744. This area is shown in Figure (It is always helpful to sketch the curve and mark the area you are determining.)

Area Under the Standard Normal Curve to the Left of $z = 1.95$

z	.00	.01	. . .	.05	. . .	.09
−3.4	.0003	.0003	. . .	.0003	. . .	.0002
−3.3	.0005	.0005	. . .	.0004	. . .	.0003
−3.2	.0007	.0007	. . .	.0006	. . .	.0005
.	.	.	. . .	.	. . .	.
.	.	.	. . .	.	. . .	.
.	.	.	. . .	.	. . .	.
1.9	.9713	.9719	. . .	.9744 ←	. . .	.9767
.	.	.	. . .	.	. . .	.
.	.	.	. . .	.	. . .	.
.	.	.	. . .	.	. . .	.
3.4	.9997	.9997	. . .	.9997	. . .	.9998

Required area



Area to the left of $z = 1.95$.

The area to the left of $z = 1.95$ can be interpreted as the probability that z assumes a value less than 1.95; that is,

$$\text{Area to the left of } 1.95 = P(z < 1.95) = \mathbf{.9744}$$

As mentioned in Section 6.1, the probability that a continuous random variable assumes a single value is zero. Therefore,

$$P(z = 1.95) = 0$$

Hence,

$$P(z < 1.95) = P(z \le 1.95) = \mathbf{.9744}$$

■ EXAMPLE

Find the area under the standard normal curve from $z = -2.17$ to $z = 0$.

Solution To find the area from $z = -2.17$ to $z = 0$, first we find the areas to the left of $z = 0$ and to the left of $z = -2.17$ in the standard normal distribution table (Table IV). As shown in Table 6.3, these two areas are .5 and .0150, respectively. Next we subtract .0150 from .5 to find the required area.

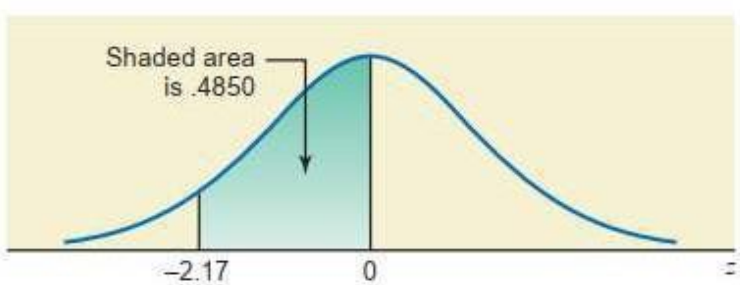
Table 6.3 Area Under the Standard Normal Curve

z	.00	.01	...	.07	...	.09
−3.4	.0003	.0003	...	.0003	...	.0002
−3.3	.0005	.0005	...	.0004	...	.0003
−3.2	.0007	.0007	...	.0005	...	.0005
.	.	.	...	.	...	.
.	.	.	...	.	...	.
−2.1	.0179	.0174	...	.0150 ←	...	.0143
.	.	.	...	.	...	.
.	.	.	...	.	...	.
0.0	.5000 ←	.5040	...	.5279	...	.5359
.	.	.	...	.	...	.
.	.	.	...	.	...	.
3.4	.9997	.9997	...	.9997	...	.9998

Area to the left of $z = 0$ — Area to the left of $z = -2.17$

The area from $z = -2.17$ to $z = 0$ gives the probability that z lies in the interval -2.17 to 0 (see Figure 6.20); that is,

$$\text{Area from } -2.17 \text{ to } 0 = P(-2.17 \le z \le 0)$$
$$= P(z \le 0) - P(z \le -2.17) = \mathbf{.5000 - .0150 = .4850}$$



Example 28

Find the following areas under the standard normal curve.

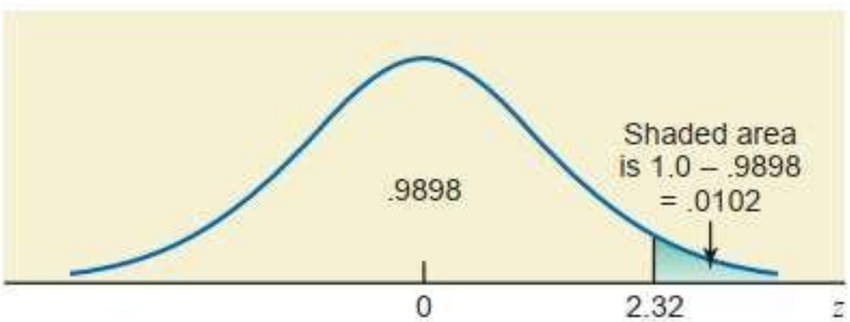
(a) **Area to the right of z=3.32**

(b) **Area to the left of z=-1.54**

Solution

(a) **As mentioned earlier, the normal distribution table gives the area to the left of a *z***

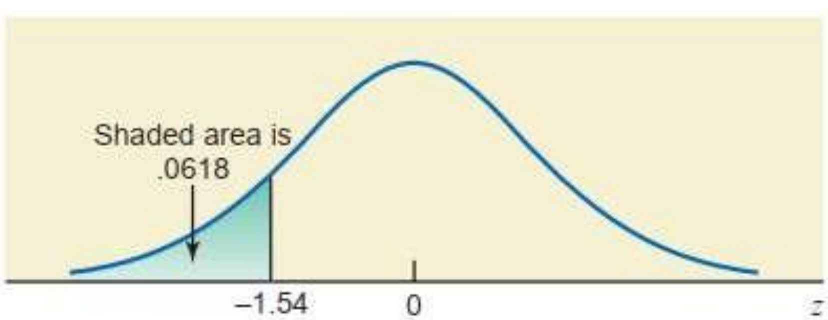
value. To find the area to the right of first we find the area to the left of Then we subtract this area from 1.0, which is the total area under the curve. From Table IV, the area to the left of is .9898. Consequently, the required area is 1.0 2 **.9898** 5 **.0102, as shown in Figure below .**



The area to the right of gives the probability that *z* is greater than 2.32. Thus,

$$\text{Area to the right of } 2.32 = P(z > 2.32) = 1.0 - .9898 = \mathbf{.0102}$$

(b) To find the area under the standard normal curve to the left of we find the area in Table IV that corresponds to in the *z* column and .04 in the top row. This area is .0618. This area is shown in Figure .



The area to the left of $z = -1.54$ gives the probability that z is less than -1.54. Thus,

$$\text{Area to the left of } -1.54 = P(z < -1.54) = \mathbf{.0618}$$

Example 29

Find the following probabilities for the standard normal curve.

(a) $P(1.19 < z < 2.12)$ **(b)** $P(-1.56 < z < 2.31)$ **(c)** $P(z > -.75)$

Solution

(a) The probability $P(1.19 < z < 2.12)$ is given by the area under the standard normal curve between $z = 1.19$ and $z = 2.12$, which is the shaded area in Figure 6.23.

To find the area between $z = 1.19$ and $z = 2.12$, first we find the areas to the left of $z = 1.19$ and $z = 2.12$. Then we subtract the smaller area (to the left of $z = 1.19$) from the larger area (to the left of $z = 2.12$).

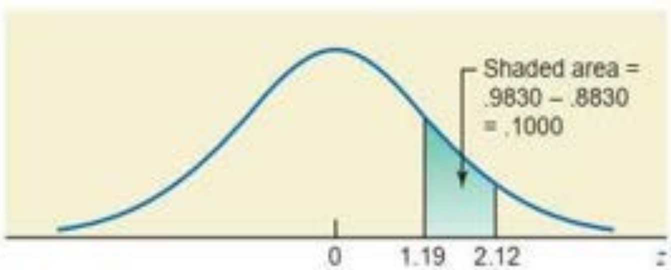


Figure 6.23 Finding $P(1.19 < z < 2.12)$.

From Table IV for the standard normal distribution, we find

$$\text{Area to the left of } 1.19 = .8830$$

$$\text{Area to the left of } 2.12 = .9830$$

Then, the required probability is

$$P(1.19 < z < 2.12) = \text{Area between } 1.19 \text{ and } 2.12$$

$$= .9830 - .8830 = \mathbf{.1000}$$

(b) The probability $P(-1.56 < z < 2.31)$ is given by the area under the standard normal curve between $z = -1.56$ and $z = 2.31$, which is the shaded area in Figure 6.24.

From Table IV for the standard normal distribution, we have

$$\text{Area to the left of } -1.56 = .0594$$

Area to the left of 2.31 = .9896

The required probability is

$$P(-1.56 < z < 2.31) = \text{Area between } -1.56 \text{ and } 2.31$$
$$= .9896 - .0594 = \mathbf{.9302}$$

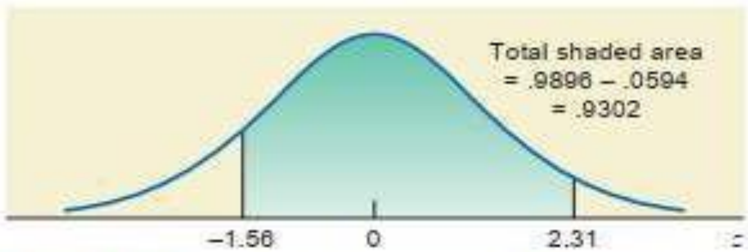


Figure 6.24 Finding $P(-1.56 < z < 2.31)$.

(c) The probability $P(z > -.75)$ is given by the area under the standard normal curve to the right of $z = -.75$, which is the shaded area in Figure 6.25.

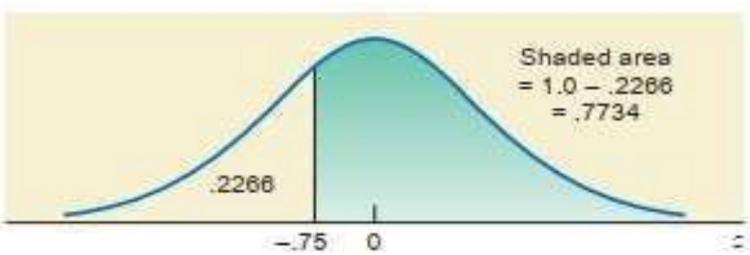


Figure 6.25 Finding $P(z > -.75)$.

From Table IV for the standard normal distribution.

$$\text{Area to the left of } -.75 = .2266$$

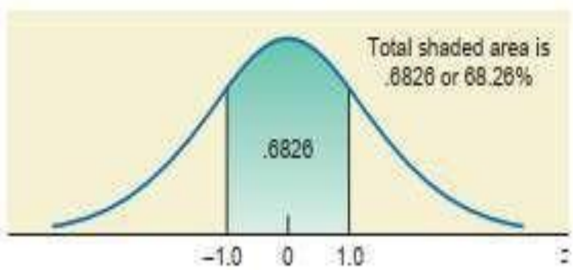
The required probability is

$$P(z > -.75) = \text{Area to the right of } -.75 = 1.0 - .2266 = \mathbf{.7734}$$

In the discussion in Section 3.4 of Chapter 3 on the use of the standard deviation, we discussed the empirical rule for a bell-shaped curve. That empirical rule is based on the standard normal distribution. By using the normal distribution table, we can now verify the empirical rule as follows.

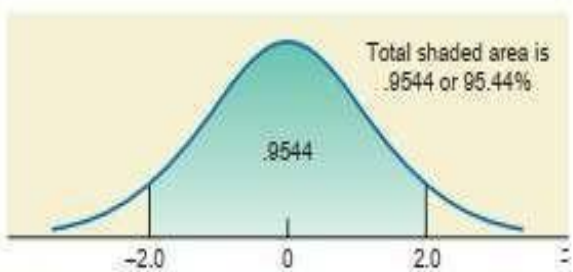
1. The total area within one standard deviation of the mean is 68.26%. This area is given by the difference between the area to the left of $z = 1.0$ and the area to the left of $z = -1.0$. As shown in Figure 6.26, this area is $.8413 - .1587 = .6826$, or 68.26%.

Area within one standard deviation of the



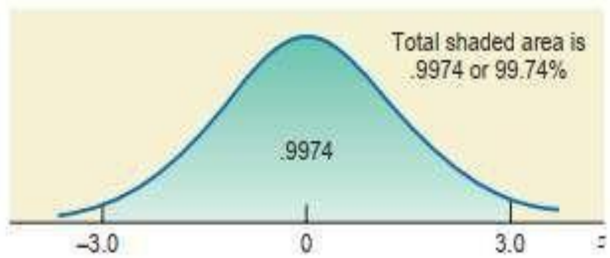
2. The total area within two standard deviations of the mean is 95.44%. This area is given by the difference between the area to the left of $z = 2.0$ and the area to the left of $z = -2.0$. As shown in Figure 6.27, this area is $.9772 - .0228 = .9544$, or 95.44%.

Area within two standard deviations of



3. The total area within three standard deviations of the mean is 99.74%. This area is given by the difference between the area to the left of $z = 3.0$ and the area to the left of $z = -3.0$. As shown in Figure 6.28, this area is $.9987 - .0013 = .9974$, or 99.74%.

Area within three standard deviations of



Again, note that only a specific bell-shaped curve represents the normal distribution. Now we can state that a bell-shaped curve that contains (about) 68.26% of the total area within one standard deviation of the mean, (about) 95.44% of the total area within two standard deviations of the mean, and (about) 99.74% of the total area within three standard deviations of the mean represents a normal distribution curve.

The standard normal distribution table, Table IV of Appendix C, goes from $z = -3.49$ to $z = 3.49$. Consequently, if we need to find the area to the left of $z = -3.50$ or a smaller value of z, we can assume it to be approximately 0.0. If we need to find the area to the left of $z = 3.50$ or a larger number, we can assume it to be approximately 1.0. Example 6–5 illustrates this procedure.

15. Exponential Distribution

the *Exponential Distribution* is another important distribution and is typically used to model times between events or arrivals. The distribution has one parameter, λ which is assumed to be the average rate of arrivals or occurrences of an event in a given time interval. its often concerned with the amount of time until some specific event occurs. For example, the amount of time (beginning now) until an earthquake occurs has an exponential distribution. Other examples include the length, in minutes, of long distance business telephone calls, and the amount of time, in months, a car battery lasts. It can be shown, too, that the value of the change that you have in your pocket or purse approximately follows an exponential distribution.

If the random variable X follows an Exponential distribution, then we write:
X~Exp(λ) The probability density function is:

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{for } x \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

where:

- **λ:** the rate parameter (calculated as λ = 1/μ)
- **e:** A constant roughly equal to 2.718

The (Cumulative) probabilities can be calculated using

$$P(X \leq x) = \begin{cases} 0 & \text{for } x < 0, \\ 1 - e^{-\lambda x} & \text{for } x \geq 0. \end{cases}$$

Example 24

The time in minutes, X, between the arrival of successive customers at a post office is exponentially distributed with pdf $f(x)=0.2e^{-0.2x}$

(A) What is the expected time between arrivals?

(B) A customer walks into the post office at 12.30 p.m. What is the probability the next customer arrives:

(i) on or before 12:32 p.m.?, (ii) after 12:35 p.m.?

Solution

A) Here $\lambda=0.2$ and so the mean time between arrivals is 1/0.2=5 minutes.

(B) (i) If the next customer arrives on or before 12:32 p.m., it means that the time between their arrival and the previous arrival is at most 2 minutes. So, we require $P(X\leq 2)$

.Using the formula above we have:

$$\begin{aligned} P(X \leq 2) &= 1 - e^{-0.2\times 2} \\ &= 1 - 0.67032 \\ &= 0.330 \text{ (to 3 d.p.)} \end{aligned}$$

(ii) If the next customer arrives after 12:35 p.m. then the time between the two customers is more than 5 minutes. We now require $P(X>5)$. To calculate $P(X>x)$. This is equivalent to $1-P(X\leq x)$ and so:

$$\begin{aligned} P(X > 5) &= 1 - P(X \leq 5) \\ &= 1 - (1 - e^{-0.2\times 5}) \\ &= e^{-1} \\ &= 0.368 \text{ (to 3 d.p.).} \end{aligned}$$

Example 25

suppose the mean number of minutes between eruptions for a certain geyser is 40 minutes. If a geyser just erupts, what is the probability that we'll have to wait less than 50 minutes for the next eruption?

To solve this, we need to first calculate the rate parameter:

- $\lambda = 1/\mu$
- $\lambda = 1/40$
- $\lambda = .025$

We can plug in $\lambda = .025$ and $x = 50$ to the formula for the CDF:

- $P(X \leq x) = 1 - e^{-\lambda x}$
- $P(X \leq 50) = 1 - e^{-.025(50)}$
- $P(X \leq 50) = 0.7135$

The probability that we'll have to wait less than 50 minutes for the next eruption is **0.7135.**