University of Technology
الجامعة التكنولوجية

Computer Science Department
قسم علوم الحاسوب

Mathematics
الرياضيات

Sora Ali Majeed
م. سرى علي مجيد

cs.uotechnology.edu.iq

Chapter One

Consider an arbitrary system of equation in unknown as:

AX = B ……………………………………………………………(1)

$$\left.\begin{array}{l} a_{r1}X_1 + a_{12}X_2 + a_B X_3 + ... + a_{1n}X_n \\ ail\chi_1 + ai_2\chi_2 + ai_3\chi 3 + ... + ain\chi n = b_1 \\ a21\chi_1 + a22\chi 2 + a23\chi 3 + ... + a2n\chi n = b2 \\ a_{21}X_1 + a_{22}X_2 + a_{23}X_3 + ... + a_{2n}X_n \\ am1\chi 1 + am2X2 + 9m3\chi 3 + ... + a2n\chi n = bm \\ a_{m1}X_1 + a_{m2}X_2 + a_{m3}X_3 + ... + a_{mn}X_n \end{array}\right\} \text{………………………(2)}$$

The coefficient of the variables and constant terms can be put in the form:

$$\begin{pmatrix} a_{11} & a_{12} & a_{1n} \\ a2 & a22 & a2n \\ a_{m1} & a_{m2} & a_{mn} \end{pmatrix}_{mxn} \begin{pmatrix} \chi_1 \\ \chi_2 \\ \\ \chi n \end{pmatrix}_{nx1} \begin{pmatrix} b1 \\ b2 \\ \\ bm \end{pmatrix}_{mx1} \text{…………………………………..(3)}$$

Let the form

$$\begin{pmatrix} a_u a_{12} a_{1n} \\ \\ a_{21} a_{22} a_{2n} \\ \\ am_1 am_2 amn \end{pmatrix} = A = (a_{i1}) \text{……………………………………………..(4)}$$

Is called (mxn) matrix and donated this matrix by:

[aij] i = 1, 2,……m and j = 1, 2,…..n.

We say that is an (mxn) matrix or تكملة

The matrix of order (mxn) it has m rows and n columns.

For example the first row is (a_{11}, a12, a_{1n})

And the first column is $\begin{pmatrix} a_{12} \\ a_{11} \\ a_{21} \\ a_{m1} \end{pmatrix}$

(aij) denote the element of matrix. Lying in the i – th row and j – th column, and we call this element as the (i,j) - th element of this matrix

Also $\begin{pmatrix} \chi 1 \\ \chi_2 \\ \\ \chi_n \end{pmatrix}_{nx1}$ is (nx1) [n rows and columns]

$\begin{pmatrix} b1 \\ b2 \\ \\ bm \end{pmatrix}_{mx}$ Is (mx1) [m rows and 1 column]

Sub – Matrix:

Let A be matrix in (4) then the sub-matrix of A is another matrix of A denoted by deleting rows and (or) column of A.

Let A = $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$

Find the sub-matrix of A with order (2×3) any sub-matrix of A denoted by deleting any row of A $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 7 & 8 & 9 \end{pmatrix}, \begin{pmatrix} 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$

Definition 1.2

If A = [aij] mxn and B = [bij] mxn are mxn matrix their sum is the mxn matrix A+B $=[aij + bij]mxn$.

In other words if two matrices have the same dimension, they may be added by addition corresponding elements. For example if:

A = $\begin{pmatrix} 2 & -7 \\ -3 & 4 \end{pmatrix}$ *and* $\mathbf{B}=\begin{pmatrix} -5 & 0 \\ 1 & 6 \end{pmatrix}$

Then

A+B = $\begin{pmatrix} 3+-5 & -7+0 \\ -3+1 & 4+6 \end{pmatrix} = \begin{pmatrix} -3 & -7 \\ -2 & 10 \end{pmatrix}$

Additions of matrices, like equality of matrices is defined only of matrices have same dimension.

Theorem 1.1**:**

Addition of matrices is commutative and associative, that is if A, B and C are matrices having the same dimension then:

A + B = B + A (commutative)

A + (b + C) = (A + B) + C (associative)

Definition 1.3

The product of a scalar K and an mxn matrix A = [aij] mxn is the nn,Xn matrix KA = [kaij] mXn for example:

$$6\begin{pmatrix}-1 & 0 & 7\\ 5 & 2 & -11\end{pmatrix}=\begin{pmatrix}6(-1) & 6(0) & 6(7)\\ 6(5) & 6(2) & 6(-11)\end{pmatrix}=\begin{pmatrix}-6 & 0 & 2\\ 30 & 12 & -6\end{pmatrix}$$

Application of Matrices

Definition 1.4**:**

If A = [aij] mxn is mxn matrix and B = [bjk] nxp an nxp matrix, the product AB is the mxp matrix C = [cik] nxp in which

$$Cik=\sum_{j=1}^{n} aij\, bik$$

Example1: if A = $\begin{pmatrix}a11 & a12 & a13\\ a21 & a22 & a23\end{pmatrix}_{2x3}$ $and\ B=\begin{pmatrix}b11\\ b21\\ b22\end{pmatrix}_{3\times1}$

A B = $\begin{pmatrix}a11\, b11+a12\, b21+a13\, b31\\ a21\, b11+a22\, b21+a23\, b31\end{pmatrix}_{2\times1}$

Example 2: Let A = $\begin{pmatrix}2 & 3\\ -1 & 4\\ 5 & -2\end{pmatrix}_{3\times2}$ $and\ B=\begin{pmatrix}3 & 1 & 4 & -5\\ -2 & 0 & 3 & 4\end{pmatrix}$

A B = $\begin{pmatrix}0 & 2 & 17 & 2\\ -11 & -1 & 8 & 21\\ 19 & 5 & 14 & -33\end{pmatrix}_{3\times4}$

Note 1.1:

1 **–** in general if A and B are two matrices. Then A B may not be equal of

BA. For example A = $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \rightarrow AB = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} and\ B\,A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$

$\therefore A\,B \neq B\,A$

2 – if A B is defined then its not necessary that B A must also be defined. For example. If A is of order (2×3) and B of order (3×1) then clearly A B is define, but B A is not defined.

1.3 Different Types of matrices:

1 – Row Matrix: A matrix which has exactly one row is called row matrix. For example (1, 2, 3, 4) is row matrix

2 – Column Matrix: A matrix which has exactly one column is called a column matrix for example $\begin{pmatrix} 5 \\ 6 \\ 7 \end{pmatrix}$ is a column matrix.

3 – Square Matrix**:** A matrix in which the number of row is equal to the number of columns is called a square matrix for example $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ is a 2×2 square matrix.

A matrix (A) (n×n) A is said to be order or to be an n-square matrix.

4 - Null or Zero Matrix**:** A matrix each of whose elements is zero is called null matrix or zero matrix, for example $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ is a (2×3) null matrix.

5 **–** Diagonal Matrix: the elements aii are called diagonal of a square matrix (a_{11} a_{22} – a_{nn}) constitute its main diagonal A square matrix whose every element other than diagonal elements is zero is called a diagonal matrix for

Example: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ or $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

6 – Scalar Matrix: A diagonal matrix, whose diagonal elements are equal, is called a scalar matrix.

For example $\begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ are scalar matrix

7 – Identity Matrix: A diagonal matrix whose diagonal elements are all equal to 1 (unity) is called identity matrix or (unit matrix). And denoted by in for

Example $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Note1.2: if A is (mxn) matrix, it is easily to define that AIn = A and also ImA = A

Ex: Find AI and IA when $A = \begin{pmatrix} 3 & 7 & 2 \\ 1 & -1 & 3 \end{pmatrix}$

Solution**:** IA $\longrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}_{2\times2} \begin{pmatrix} 3 & 7 & 2 \\ 1 & -4 & 3 \end{pmatrix}_{2\times3} = \begin{pmatrix} 3 & 7 & 2 \\ 1 & -4 & 3 \end{pmatrix}_{2\times3}$

And AI = $\begin{pmatrix} 3 & 7 & 2 \\ 1 & -4 & 3 \end{pmatrix}_{2\times3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{3\times3} = \begin{pmatrix} 3 & 7 & 2 \\ 1 & -4 & 3 \end{pmatrix}_{2\times3}$

8 – Triangular Matrix**:** A square matrix (aij) whose element aij = 0 whenever $j \langle i$ is Called a lower triangular matrix.simillary y a square matrix (aij) whose element aij = 0 whenever is called an upper Tringular Matrix

For example: $\begin{pmatrix} 1 & 0 & 0 \\ 4 & 5 & 0 \\ 7 & 8 & 9 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$ are lower triangular matrix

And

$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix}$, $\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$, are upper triangular

Definition 1.4:

Transpose of matrix

The transpose of an mxn matrix A is the nxm matrix denoted by $A^{\mathbf{T}}$, formed by interchanging the rows and columns of A the ith rows of A is the ith columns in $A^{\mathbf{T}}$**.**

For Example**:** A = $\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & -1 \end{pmatrix}_{2\times3}$ $A^T = \begin{pmatrix} 1 & 0 \\ 2 & 1 \\ 3 & -1 \end{pmatrix}_{3\times2}$

9 – Symmetric Matrix: A square matrix A such that A = $A^{\mathbf{T}}$ is called symmetric matrix i.e. A is a symmetric matrix if and only if aij = a_{ji} for all element.

$$\begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{pmatrix}$$

For Example: (a) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$, (b)

10 – Skew symmetric Matrix: A square matrix A such that A = A^T is called that A is skew symmetric matrix. i.e A is skew matrix $\longleftrightarrow$ a_{ji} = -aij for all element of A.

The following are examples of symmetric and skew – symmetric matrices respectively

$$(a)\begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{pmatrix},(b)\begin{pmatrix} 0 & 1 & 2 \\ -1 & 0 & 3 \\ -2 & -3 & 0 \end{pmatrix}$$

(a) symmetric

(b) Skew – symmetric.

Note the fact that the main diagonal element of a skew – symmetric matrix must all be Zero

11 – Determinates: To every square matrix that is assigned a specific number called the determinates of the matrix.

(a) Determinates of order one: write det (A) or $|A|$ for detrimental of the matrix A. it is a number assigned to square matrix only.

The determinant of (1×1) matrix (a) is the number a itself det (a) = a.

(c) Determinants of order two: the determinant of the 2×2. matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Is denoted and defined as follows: $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

Theorem 1.2: determinant of a product of matrices is the product of the determinant of the matrices is the product of the determinant of the matrices det (A B) = det (A). det (B) det (A + B) # det a + det B

(C) Determinates of order three:

(i) the determinant of matrix is defined as follows:

$$\begin{vmatrix} + & - & + \\ a111 & a22 & a23 \\ a21 & a22 & a23 \\ a31 & a32 & a33 \end{vmatrix} = a11 \begin{vmatrix} a22 & a23 \\ a32 & a33 \end{vmatrix} - a12 \begin{vmatrix} a21 & a23 \\ a31 & a33 \end{vmatrix} + a13 \begin{vmatrix} a21 & a22 \\ a31 & a32 \end{vmatrix}$$

(ii) Consider the (3×3) matrix $\begin{vmatrix} a11 & a12 & a13 \\ a21 & a22 & a23 \\ a31 & a32 & a33 \\ & & \end{vmatrix}$

= a11 a22 a33 + a21 a22 a31 + a13 a21 a32

\- a31 a22 a13 - a32 a23 a11 -a33 a21 a12.

Show that the diagram papering below where the first two columns are rewritten to the right of the matrix.

Theorem 1.3:

A matrix is invertible if and only if its determinant is <u>not Zero</u> usually a matrix is said to be singular if determinant is zero and non singular it otherwise.

1.6 Rank of Matrix: we defined the rank of any matrix a that the order of the largest square sub-matrix of a whose determinant not zero (det of sub-matrix ‡ 0)

Example: Let A = $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$ find the rank of A

$1 \times 9 \times 5 + 2 \times 6 \times 7 + 3 \times 4 \times 8 - 3 \times 5 \times 7 - 1 \times 6 \times 8 - 2 \times 4 \times 9 = 0$

Since $|A|$ of order 3   Rank ‡ 3

Since $\begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix} = -3 \neq 0$ the rank ‡ 2

1.7 Minor of matrix**:** Let A = $\begin{pmatrix} a11 & a12 & a1n \\ a21 & a22 & a2n \\ & & \\ an1 & an2 & ann \end{pmatrix}$ (4)

Is the square matrix of order n then the determinant of any square sub-matrix of a with order (n-1) obtained by deleting row and column is called the minor of A and denoted by Mij.

1.8 Cofactor of matrix: Let A be square matrix in (4) with mij which is the minors of its. Then the Cofactor of a defined by Cij = $(-1)^{\mathbf{i+j}}$ Mij

Example: Let A = $\begin{pmatrix} -2 & 4 & 1 \\ 4 & 5 & 7 \\ -6 & 1 & 0 \end{pmatrix}$ find the minor and the cofactor of element 7.

Solution: The minor of element 7 is

M23 = det $\begin{pmatrix} -2 & 4 \\ -6 & 1 \end{pmatrix} = \begin{vmatrix} -2 & 4 \\ -6 & 1 \end{vmatrix} = 22$

i.e (denoted by take the square sub-matrix by deleting the second rows and third column in A).

the Cofactor of 7 is

$$C23 = (-1)^{2+3} M_{23} = (-1)^{2+3} \begin{vmatrix} -2 & 4 \\ -6 & 1 \end{vmatrix} = -22$$

1.9 Adjoint of matrix: Let matrix A in (4) then the transposed of matrix of cofactor of this matrix is called adjoint of A, adjoint A = transposed matrix of Cofactor.

The inverse of matrix**:** Let A be square matrix. Then inverse of matrix {Where A is non-singular matrix} denoted by A^{-1} and $A^{-1} = \frac{1}{\det A} adj(A)$

1.0 method to find the inverse of A**:** To find the inverse of matrix we must find the following:

(i) the matrix of minor of elements of A.

(ii) the Cofactor of minor of elements of A

(iii) the adjoint of A .

then $A^{-1} = \frac{1}{|A|} adjA$

Example: let A = $\begin{pmatrix} 2 & 3 & -4 \\ 1 & 2 & 3 \\ 3 & -1 & -1 \end{pmatrix}$ Find A^{-1}

(1) Minors of A is Mij = $\begin{pmatrix} 1 & -10 & -7 \\ -7 & 10 & -11 \\ 17 & 10 & 1 \end{pmatrix}$

(2) Cofactor of A is (-1) Mij = $\begin{pmatrix} 1 & 10 & -7 \\ 7 & 10 & 11 \\ 17 & -10 & 1 \end{pmatrix}$

(3) Adj of A = $\begin{pmatrix} 1 & 7 & 17 \\ 10 & 10 & -10 \\ -7 & 11 & 1 \end{pmatrix}$.

(4) det = 60

$$A^{-1} = \frac{1}{60}\begin{pmatrix} 1 & 7 & 17 \\ 10 & 10 & -10 \\ -7 & 11 & 1 \end{pmatrix}.$$

1.11 Properties of Matrix Multiplication:

1 – (KA) B = K (AB) = A (KB)   K is any number

2 – A (BC) = (AB) C

3 – (A + B) C = AC + BC

4 – C (A + B) = CA + CB

5 – AB ‡ BA (in general)

For example: Let A = $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and B = $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$

$$A\,B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$B\,A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

A B =‡ B A

6 – A B = 0 but not necessarily A = 0 or B = 0

For Example: A = $\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}, B = \begin{pmatrix} -1 & +1 \\ +1 & -1 \end{pmatrix}$

$$A\,B = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$A \neq 0, B \neq 0$$

But

A B = 0

7 - $\begin{pmatrix} C & 0 & 0 \\ 0 & C & 0 \\ 0 & 0 & C \end{pmatrix} = C\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

8 – A I = I A = A   where I is identity matrix

Chapter Two

Function

Numbers:

1 – N = set of natural numbers

N = {1, 2, 3, 4……}

2 – I = set of integers

= {……., -3, -2, -1, 0, 1, 2, 3…}

Note that: NCI

3 – A = set of rational numbers

$= \left(\chi : \chi = \frac{\rho}{q} \;\; \rho \, and \, q \, are \, \text{int} \, egers \, q \neq 03 \right]$

Ex: $\frac{3}{2}, -\frac{4}{5}, \frac{3}{1}, \frac{-7}{1}$

Note that: ICA

4 – B = set of irrational numbers

= $\{X : X \, is \, not \, arational \, number\}$

Ex: $\sqrt{2}$, $\sqrt{3}$, $-\sqrt{7}$

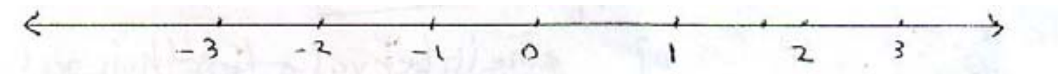
5 – R: set of real numbers

= set of all rational and irrational numbers

Note that

R = AUB

Note: the set of real numbers is represented by a line called a line of numbers:



(ii) NCR, ICR, ACR, BCR

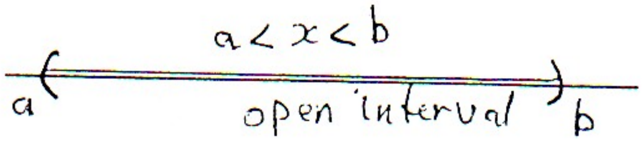
Intervals

The set of values that a variable χ may take on is called the domain of χ. The domains of the variables in many applications of calculus are intervals like those shows below.

- **open intervals**

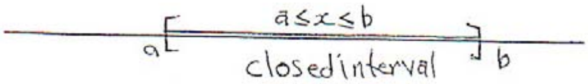
is the set of all real numbers that lie strictly between two fixed numbers a and b:

In symbols	**In words**
$a \langle \chi \langle b \, or \, (q, b)$	The open interval a b



- Closed Intervals contain both endpoints:

In symbols	**In words**
$a \le \chi \le b \quad or \quad [a,b]$	the closed interval a b

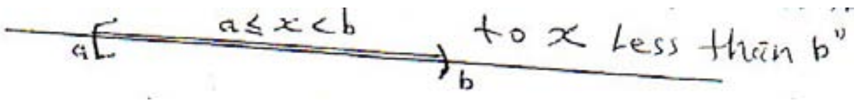


- Half – open intervals contain one but not both end points:

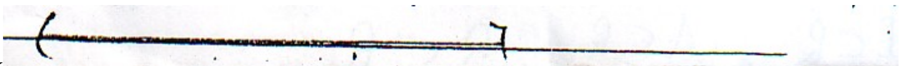
In symbols: **in wards**

$a \le \chi \langle b$ or [a,b] ‘ the interval a less than or equal

$a \le \chi c b$ To χ less than b



$a \langle \chi \le b$ or [a,b] the interval a less than χ less than or equal b



Ex. find the domain of

1 – Y = $\sqrt{1-X^2}$

The domain of χ is the closed interval

$1- \le \chi \le 1$

2 – Y = $\dfrac{1}{\sqrt{1-X^2}}$

The domain for χ is open interval

$-1 \langle \chi \langle 1$ because $\dfrac{1}{0}$ is not defined

B - y = $\sqrt{\dfrac{1}{X}-1}$

$1/X - 1 \ge 0 \quad or \quad \dfrac{1}{X} \ge 1$

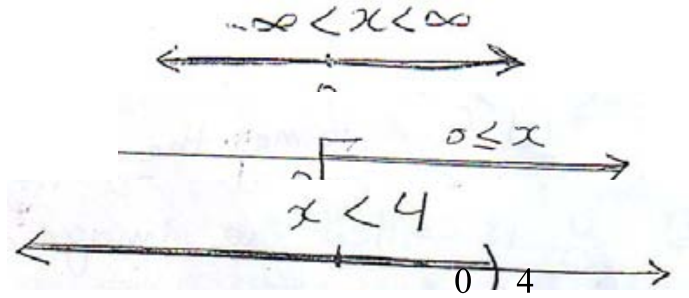
The domain for χ is the half – open $0 < \chi \leq 1$

Ex: **the equation** **the domain**

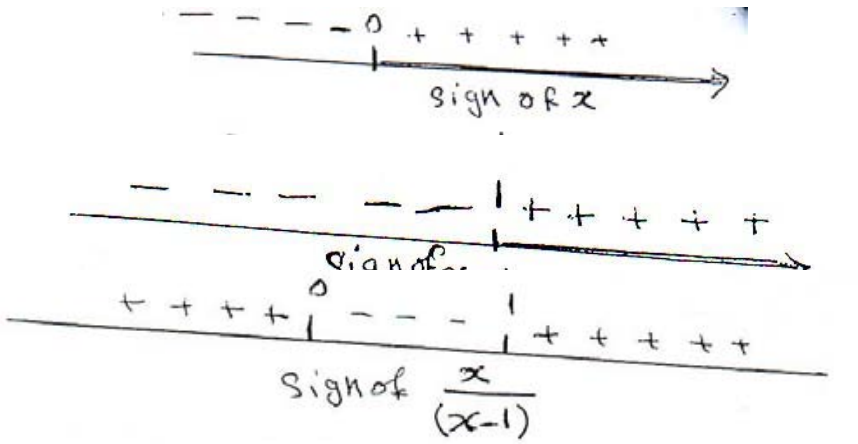
Y = χ^2

Y = $\sqrt{\chi}$

Y = $\frac{1}{\sqrt{4-X}}$



$y = \sqrt{\frac{\chi}{(\chi - 1)}}$



The domain for χ is $X \leq 0 \, U \, X > 1$

Definition: A function, say f is a relation between the elements of two sets say A and B such that for every $\chi \in A$ there exists one and only one $Y \in B$ with Y = F(X).

The set A which contain the values of χ is called the domain of function F.

The set B which contains the values of Y corresponding to the values of χ is called the range of the function F. χ is called the independer variable of the function F, while Y is called the dependant variable of F.

Note:

1 – Some times the domain is denoted by DF and the range by RF.

2 – Y is called the image of χ.

Example: Let the domain of χ be the set $\{0,1,2,3,4\}$. Assign to each value of χ the number Y = χ^2. The function so defined is the set of pairs, { (0,0), (1,1), (2,4), (3,9), (4,16) }.

Example: Let the domain of χ be the closed interval $-2 \le \chi \le 2$. Assign to each value of χ the number y = χ^2.

The set of order pairs (χ, y) such that $-2 \le \chi \le 2$

And y = χ^2 is a function.

Note: Now can describe function by two things:

1 – the domain of the first variable χ.

2 – the rule or condition that the pairs (χ, y) must satistfy to belong to the function.

Example:

The function that pairs with each value of χ diffrent from 2 the number

$$\frac{\chi}{\chi - 2}$$

$$y = f(\chi) = \frac{\chi}{\chi - 2} \quad \chi \ne 2$$

Note 2: Let $f(\chi)$ and $g(\chi)$ be two function.

1 - $(f \pm g)(\chi) = f(\chi) \pm g(\chi)$

2 - $(f.g)(\chi) = f(\chi) \cdot g(\chi)$

3 - $(\frac{f}{g})(\chi) = \frac{f(\chi)}{g(\chi)}$ if g $(\chi) \ne 0$

Example: Let $f(\chi) = \chi + 2, g(\chi) = \sqrt{\chi - 3}$ evaluate

$$f \pm g, f.g \; and \; \frac{f}{g}$$

So: $(f \pm g)(\chi) = f(\chi) \pm g(\chi) = \chi + 2 \pm (\sqrt{\chi - 3}$

$(f.g)(\chi) = f(\chi) \, . \, g(\chi) = (\chi + 2)(\sqrt{\chi - 3}$

$(\frac{f}{g})(\chi) = \frac{f(\chi)}{g(\chi)} = \frac{\chi + 2}{\sqrt{\chi - 3}} \quad \{X : X \rangle 3\}$

Composition of Function:

Let $f(\chi)$ and $g(\chi)$ be two functions

We define: $(fog)(\chi) = f(g(\chi))$

Example: Let $f(\chi) = \chi^2, \, g(\chi) = \chi - 7$ evaluate fog and gof

So: $(fog)(\chi) = f[g(x)] = f(\chi - 7) = (\chi - 7)^2$

$(gof)(\chi) = g[f(\chi)] = g(\chi^2) = \chi^2 - 7$

$\therefore fog \neq gof$

Inverse Function

Given a function F with domain A and the range B.

The inverse function of f written f, is a function with domain B and range A such that for every $y \in B$ there exists only $\chi \in A$ with $\chi = f^{-1}(y)$.

Note that: $f^{-1} \neq \frac{1}{f}$

Polynomials: A polynomial of degree n with independent variable, written $\mathbf{f_n(x)}$ or simply $f(\chi)$ is an expression of the form:

$fn(\chi) = q_o + a_1 \chi + a_2 X^2 + ... + anX^n$ ....(*)

Where $q_0, a1, ..., an$ are constant (numbers).

The degree of polynomial in equation (*) is n (the highest power of equation)

Example:

(i) f (χ) = 5X polynomial of degree one.

(ii) f $(\chi) = 3X^5 - 2X + 7$ polynomial of degree five.

(iii) F (χ) = 8 polynomial of degree Zero.

Notes:

The value of χ which make the polynomial f (χ) = 0 are called the roots of the equation $(f(\chi) = 0)$

Example: (χ) = 2 is the root of the polynomial

F $(\chi) = \chi^2 - \chi - 2$

Since f (2) = 0

Example: F (χ) Linear function if

F $(\chi) = a\chi + b$.

Even Function:

F (χ) is even if f (-x) = F (x)

Example: 1 - F $(\chi) = (\chi)^2$ is even since $f(-\chi) = (-\chi)^2 = (\chi)^2 =$ f (χ)

2 - F (χ) = cos (χ) is even because $f(-\chi)$ = cos $(-\chi)$ = cos (χ) = f (χ)

Odd Function:

If $f(-\chi) = -f(\chi)$ the function is called odd.

Example: 1 - $f(\chi) = \chi^3$ is odd since $f(-\chi) = -\chi 3 = -f(\chi)$

2 - $f(\chi) = Sin(-\chi) = -Sin\,X = -f(\chi)$.

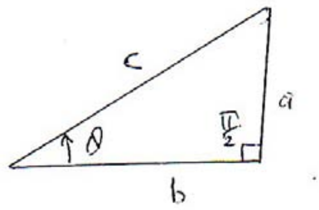
Trigonometric Function:

1 – Sin $\varphi = \frac{a}{c}$

$2 - \text{Cos}\ \ \varphi = \frac{b}{c}$

$3 - \tan\ \ \varphi = \frac{a}{b}$

$4 - \text{Cotan}\ \varphi = \frac{1}{\tan\varphi} = \frac{b}{a}$

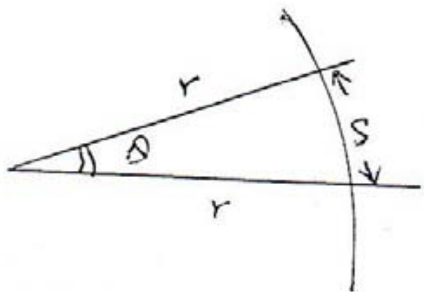


$5 - \text{Sec}\ \ \varphi = \frac{1}{\cos\varphi} = \frac{c}{b}$

$6\text{- CSC}\ \ \varphi = \frac{1}{Sin\,\varphi} = \frac{c}{a}$

Relation ships between degrees and radians

φ In radius = $\frac{s}{r}$



$360^{\mathbf{o}} = \frac{2\pi r}{r}$

$= 2\,\pi\, radius$

$1^{\mathbf{o}} = \frac{\pi}{180}$ radius = 0.0174 radian

1 radian = $\frac{180}{\pi} \deg ree = 57.29578^{\circ}$

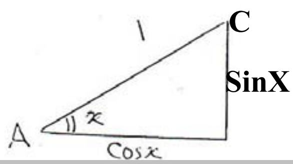
$\left(\frac{360}{2\,\pi}\right) = 1 radian = 57^{\circ}.18$

$180^{\mathbf{o}} = \pi$ radians = 3.14159 – radians

$1^{\mathbf{o}} = \frac{2\,\pi}{360} = \frac{\pi}{180} \approx 0.001754$ radians

$\tan\chi = \frac{\sin\chi}{\cos\chi}$

$Cot\chi = \frac{Cos\chi}{Sin\chi} = \frac{1}{\tan\chi}$



B

$$Sec\,\chi = \frac{1}{\cos\chi}$$

$$Csc\ \chi = \frac{1}{\sin\chi}$$

$$Cos^2\,\chi + Sin^2\,\chi = 1$$

$$\tan^2\chi + 1 = Sec^2\chi$$

$$Cot^2\,\chi + 1 = Csc^2\,X$$

$$Sin(\chi \pm y) = Sin \times Cosy \mp Cos \times Sin\,y$$

$$Cos(\chi \pm y) = Cos \times Cosy \pm Sin \times Siny$$

$$\tan(\chi \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$$

1 − $SinA + Sin\,B = 2\,Sin\,\frac{A+B}{2}\,Cos\,\frac{A-B}{2}$

2 - $Sin\,A - Sin\ B = 2\,Cos\,\frac{A+B}{2}\,Sin\,\frac{A-B}{2}$

3 − $Cos\,A + Cos\ b = 2\,Cos\,\frac{A+B}{2}\,Cos\,\frac{A-B}{2}$

4 - $Cos\,A - Cos\,B = 2\,Sin\,\frac{A+B}{2}\,Sin\,\frac{A-B}{2}$

$$Sin\,2\,X = 2\,Sin\ X\ CosX$$

$$Cos^2 = Cos^2 X - Sin^2 X$$

$$= 1 - 2\,Sin^2 X$$

$$= 2Cos^2 X - 1$$

$$Cos^2 x = \frac{1 + Cos^2 x}{2}$$

$$Sin^2\ x = \frac{1 - Cos^2 x}{2}$$

$$Sin(\varphi + 2\pi) = Sin\,\varphi$$

$$Cos(\varphi + 2\pi) = Cos\,\varphi$$

$$\tan(\varphi + \pi) = \tan\varphi$$

Degree	O^{o}	30^{o}	45^{o}	60^{o}	90^{o}	180^{o}	270^{o}	360^{o}
θ radius	O	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
Sin θ	O	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\sqrt{\frac{3}{2}}$	1	O	-1	O
Cos θ	1	$\sqrt{\frac{3}{2}}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	O	-1	O	1
tan θ	O	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$				

$$Cos(\varphi + 2n\pi) = Cos\,\varphi$$
$$Sin\ (\varphi + 2n\pi) = Sin\,\varphi$$
$$Cos(-\varphi) = Cos\,\varphi$$
$$Sin\ (-\varphi) = -Sin\,\varphi$$
$$Cos(\frac{\pi}{2} + \varphi) = Cos\,\varphi$$
$$\tan\ (\pi - \varphi) = -\tan\varphi$$
$$\tan\ (\frac{\pi}{2} + \varphi) = -Cot\varphi$$

Chapter Three

The derivative

1 – Derivative of a function:

Let = $y = f(X)$ and let P(X1, y1) be fixed point on the curve, and Q $(X1 + \Delta X, y1 + \Delta y)$ is another point on the curve as see in the figure

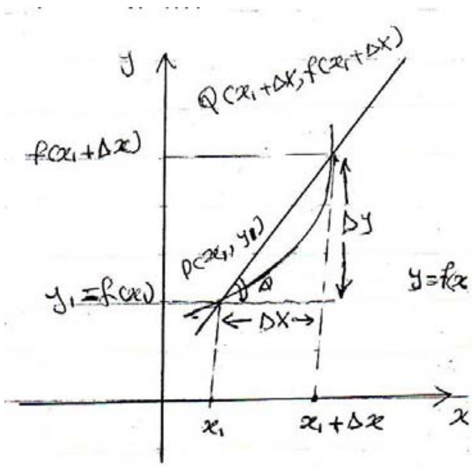
$y1 = f(X1)$, and

$y1 + \Delta y = f(X1 + \Delta X)$

$\Delta y = f(X1 + \Delta X) - y1$

Divided by ΔX

$$\frac{\Delta y}{\Delta X} = \frac{f(X_1 + \Delta X) - f(X_1)}{\Delta X}$$



The

slope of the curve $f(X)$ is

M = $\tan\phi = \frac{\Delta y}{\Delta X}$

$$\therefore M = \frac{f(X + \Delta X) - f(X)}{\Delta X}$$

We define the limit may exist for some value of X_1 .

At each point χ_1 where limit does exist, then f is said to have a derivative or to be differentiable.

Rules of Derivations:

$y = f(X) = C$ C constant

$y^1 = f^1(X) = \frac{dy}{dX} = 0$

$- f(X) = X^n$

$f^1(X) = nX^{n-1}$

$- f(X) = CnX^{n-1}$

$- f(X) = U \pm V$ n Positive integer

$f^1(X) = \frac{dy}{dX} = \frac{du}{dx} + \frac{dv}{dX}$

$- f(X) = UV$

$f^1(X) = U\frac{dv}{dX} + V\frac{du}{dX}$

$- f(X) = \frac{u}{v}$

$f^1(X) = \frac{vu^1 - uv^1}{V^2}$, $Where\ U^1 = \frac{du}{dX}$

$- f(X) = [U]^n$

$f^1(X) = n[U]^{n-1}\frac{du}{dX}$

$- f(X) = \overset{u}{e}$

$f^1(X) = \overset{u}{e}\frac{du}{dX}$

$- f(X) = \overset{U}{C}$ $U\ C\ Cons\tan t$

$f^1(X) = \overset{U}{C}.LnC\ .\frac{du}{dx}$

Derivative of trigonometric functions:

1) $(\sin u)' = \cos u\, du$
2) $(\cos u)' = -\sin u\, du$
3) $(\tan u)' = \sec^2 u\, du$
4) $(\cot u)' = -\csc^2 u\, du$
5) $(\sec u)' = \sec u \tan u\, du$
6) $(\csc u)' = -\csc u \cot u\, du$

derivative of the inverse trigonometric functions:

1) $(\sin^{-1} u)' = du/(1-u^2)^{1/2}$
2) $(\cos^{-1} u)' = -du/(1-u^2)^{1/2}$
3) $(\tan^{-1} u)' = du/1+u^2$
4) $(\cot^{-1} u)' = -du/1+u^2$
5) $(\sec^{-1} u)' = du/\ u(u^2-1)^{1/2}$
6) $(\csc^{-1} u)' = -du/\ u(u^2-1)^{1/2}$

ex: find y' of

(1) $y = [\ln(3x+1)]^3$ (2) $y = 4^x$

Sol:

(1) $y' = 3[\ln(3x+1)]^2 [3/(3x+1)] = 9[\ln(3x+1)]^2/(3x+1)$

(2) $y' = 4^x \ln 4$

(2) $y = x \sin^{-1} x + (1 - x^2)^{1/2}$

(3) $y = \cosh^{-1}(\sec x)$

sol:

(1) $y' = \sec^2(3x^2)\, 6x = 6x \sec^2(3x^2)$

(2) $y' = x/(1 - x^2)^{1/2} + \sin^{-1} x - x/(1 - x^2)^{1/2} = \sin^{-1} x$

(3) $y' =[1/(\sec^2 x - 1)^{1/2}] \sec x \tan x = \sec x \tan x/(\sec^2 x - 1)^{1/2}$

(Taylor's series): If a function f can be represented by a power series in (x-b) called Taylor's series and has the form:

$$f(x) = f(b) + f^{'}(b)(x-b) + \frac{f^{''}(b)(x-b)^2}{2!} + ... + \frac{f^{n}(b)(x-b)^n}{n!} + ...$$

<u>Example:</u>

Find Taylor series expansion of cos x about a point a=2π

Sol:

f(x) = cos x

f(2π) = cos (2π) = 1

$f^{'}(x) = -\sin x,\ f^{'}(2\pi) = -\sin 2\pi = 0$

$f^{''}(x) = -\cos x,\ f^{''}(2\pi) = -\cos 2\pi = -1$

$f^{'''}(x) = \sin x,\ f^{'''}(2\pi) = \sin 2\pi = 0$

$f^{iv}(x) = \cos x,\ f^{iv}(2\pi) = \cos 2\pi = 1$

$$\cos x = 1 - \frac{(x-2\pi)^2}{2!} + \frac{(x-2\pi)^4}{4!} - \frac{(x-2\pi)^6}{6!} + ...$$

(Maclaurin series): when b = 0, Taylor series called Maclaurin series.

Example:

Find Maclaurin series for the function f(x) = e^x

Sol:

f(x) = $e^x \rightarrow$ f(0) = e^0 = 1

$f^{'}(x) = e^x \rightarrow f^{'}(0) = e^0 = 1$

$f^{''}(x) = e^x \rightarrow f^{''}(0) = e^0 = 1$

$f^{'''}(x) = e^x \rightarrow f^{'''}(0) = e^0 = 1$

$e^x = 1 + x + (x^2/2!) + (x^3/3!) + ...$

Chapter Four

(INTEGRALS)

The process of finding the function whose derivative is given is called integration, it's the inverse of differentiation.

Definition:(indefinite integral)

A function $y=F(x)$ is called a solution of $dy/dx=f(x)$ if $dF(x)/dx=f(x)$.

We say that $F(x)$ is an integral of $f(x)$ with respect to x and $F(x) + c$ is also an integral of $f(x)$ with a constant c s.t

$D(F(x) +c)=f(x)$.

Formulas of Integration:

1) $\int dx=x + c$.
2) $\int a\, dx=a \int dx$
3) $\int(du \pm dv)= \int du \pm \int dv$.
4) $\int x^n\, dx=(x^{n+1}/n+1)+ c$
5) $\int (u)^n\, du=(u^{n+1}/n+1)+ c$
6) $\int e^u\, du= e^u+ c$
7) $\int a^u\, du=(a^u/\ln a) + c$
8) $\int du/u= \ln u + c$.

Example1:

Solve the differential equation: $dy/dx=3x^2$.

Sol:

$dy=3x^2\, dx$

since $d(x^3)=3x^2\, dx$, then we have:

$\int dy= \int 3x^2\, dx= \int d(x^3)\, dx$

$y= x^3+ c$.

9 methods for finding integrals:

1"Integral of trigonometric functions":

1) $\int \cos u \, du = \sin u + c$
2) $\int \sin u \, du = -\cos u + c$
3) $\int \sec^2 u \, du = \tan u + c$
4) $\int \csc^2 u \, du = -\cot u + c$
5) $\int \sec u \tan u \, du = \sec u + c$
6) $\int \csc u \cot u \, du = -\csc u + c$

Integral of the inverse trigonometric functions:

1) $\int du/(1-u^2)^{1/2} = \{\sin^{-1}u + c \quad \text{or} \quad -\cos^{-1}u + c\}$
2) $\int du/1+u^2 = \{\tan^{-1}u + c \quad \text{or} \quad -\cot^{-1}u + c\}$
3) $\int du/\, u(u^2-1)^{1/2} = \sec^{-1}u + c \quad \text{or} \quad -\csc^{-1}u + c\}$

ex:

evaluate:

1) $\int (5x^4-6x^2+2/ x^2)dx$

2) $\int \cos 2x \, dx$

3) $\int \cos^2 x \, dx$

sol:

1) $\int (5x^4-6x^2+2/ x^2)dx = 5\int x^4dx-6\int x^2dx+2\int x^{-2}dx$

$X^5 -2x^3-2/x+ c$

2) $\int \cos 2x \, dx = \sin 2x/2 + c$

3) $\int \cos^2 x \, dx = 1/2 \int(1+ \cos 2x) \, dx = 1/2[\int dx + \int \cos 2x \, dx]$

$1/2[x + \sin 2x/2] + c = 1/2x + 1/4 \sin 2x + c$

3- "Integration by parts"

Let u and v be functions of x and d(uv) = u dv + v du

By integration both sides of this equation (w.r.t x)

$\int d(uv)=\int u \, dv +\int v \, du$ this implies $(uv)=\int u \, dv +\int v \, du$

$\int u \, dv= (uv) - \int v \, du$

ex: find $\int x e^x dx$

sol:

let $u=x \rightarrow du = dx$ and let $dv = e^x dx$

from $\int u \, dv= (uv) - \int v \, du \rightarrow \int x e^x dx= x e^x- e^x +c$

4- "Integrals involving $(a^2 - u^2)^{1/2}$, $(a^2 + u^2)^{1/2}$, $(u^2 - a^2)^{1/2}$, $a^2 - u^2$, $a^2 + u^2$, $u^2 - a^2$ "

(A) $u= a \sin \Phi$ replaces $a^2 - u^2 = a^2 - a^2 \sin^2 \Phi = a^2(1- \sin^2 \Phi) = a^2\cos^2 \Phi$

(B) $u= a \tan \Phi$ replaces $a^2 + u^2 = a^2 + a^2 \tan^2 \Phi = a^2 \sec^2 \Phi$

(C) $u= a \sec \Phi$ replaces $u^2 - a^2 = a^2 \sec^2 \Phi - a^2 = a^2 \tan^2 \Phi$

Ex: find $\int dx/x^2(4 - x^2)^{1/2}$

Sol:

Let $x=2\sin \Phi \rightarrow dx = 2\cos \Phi \, d\Phi$

$\int dx/x^2(4 - x^2)^{1/2} = \int 2\cos \Phi \, d\Phi/4 \sin^2 \Phi(4- 4 \sin^2 \Phi)^{1/2} =$

$\int 2\cos \Phi \, d\Phi/4 \sin^2 \Phi(2\cos \Phi) = \int d\Phi/4 \sin^2 \Phi = 1/4\int \csc^2 \Phi \, d\Phi = -1/4\cot \Phi + c$

now

from $x= 2\sin \Phi \rightarrow \sin \Phi = x/2 \rightarrow \cos \Phi = (1- x^2/4)^{1/2} = 1/2(4 - x^2)^{1/2}$

$-1/4\cot \Phi + c = (-1/4)(4 - x^2)^{1/2}/x$

6-"method of partial fractions"

If the integral of the form f(x)/g(x) s.t f(x) and g(x) are poly.

And degree of f(x)< degree of g(x) we can carry out two cases:

Case i

If all factor of g(x) are linear, by the following ex:

Ex: find $\int dx/ x^2 + x - 2$

Sol:

$1/ x^2 + x - 2 = 1/(x-1)(x+2) = A/(x-1) + B/(x + 2) =$

$[A(x + 2) + B/(x - 1)] /(x-1) (x + 2)$

$1= Ax +A2 +Bx - B = (A+B)x + (A2-B)$

$1=A2-B$

$0= (A+B)$

$3A=1 \rightarrow A=1/3$ put in eq.(2) $\rightarrow B =-1/3$

$\int dx/ x^2 + x - 2 = \int 1/(x-1)(x+2)dx = \int [A/(x-1) + B/(x + 2)]dx$

$= \int [(1/3)/(x-1) - (1/3)/(x + 2)]dx = 1/3\int /(x-1) - 1/3 \int /(x + 2)dx$

$= 1/3 \ln|x-1| -1/3 \ln|x+2|+c$

Case ii

If some of the factors of g(x) are quadratic, by the following ex:

Ex: find $\int (x^2 + x - 2)dx/ (3x^3- x^2 + 3x - 1)$

Sol:

$(x^2 + x - 2) / (3x^3- x^2 + 3x - 1) = (x^2 + x - 2)/ x^2(3x - 1) + (3x - 1)$

$= (x^2 + x - 2)/ (3x - 1) (x^2+1) = [A/(3x - 1)] +[(Bx + C)/ (x^2+1)]$

$= [A(x^2+1)+ (Bx + C) (3x - 1)]/ (3x - 1) (x^2+1)$

$x^2 + x - 2 = A(x^2+1)+ (Bx + C) (3x - 1)$

$x^2 + x - 2 = (A + 3B) x^2 + (B + 3C) x + (A-C)$

$A + 3B= 1$

$B + 3C = 1$

<u>$A - C = -2$</u>

$A = -7/5$, $B = 4/5$, $C = 3/5$

$(x^2 + x - 2)/ (3x - 1) (x^2+1) = (-7/5)/(3x - 1) + [(4/5)x + (3/5)]/ (x^2+1)$

And

$\int (x^2 + x - 2)dx/ (3x - 1) (x^2+1) = (-7/5) \int dx/(3x - 1) +$

$(4/5) \int x\, dx/(x^2+1) + (3/5) \int dx/ (x^2+1)$

$= -(7/15) \ln |3x-1| + (2/5) \ln | x^2+1| + 3/5 \tan^{-1} x$

9-"evaluating integrals of the following types"

(A) $\sin(mx)\sin(nx) = (1/2)\,[\cos(m-n)x - \cos(m+n)x]$

(B) $\sin(mx)\cos(nx) = (1/2)\,[\sin(m-n)x + \sin(m+n)x]$

(C) $\cos(mx)\cos(nx) = (1/2)\,[\cos(m-n)x + \cos(m+n)x]$

Ex:

$\int 2\sin(4x)\sin(3x)\,dx = \int (2/2)\,[\cos(4-3)x - \cos(4+3)x]dx$

$= \int (\cos x - \cos 7x)dx = \sin x - (1/7)\sin 7x + c$

CHAPTER Five

Definition

1-Partial Derivative

If f is a function of the variables x, and y in the region xy plane the **Partial Derivative** of f with respect to (w. r. to) x, at point (x, y) is

$$\partial f/\partial x = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}$$

And (w. r. to) y at point (x, y) is

$$\partial f/\partial y = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x,y)}{\Delta y}$$

To find $\partial f/\partial x$ is simply regards y as constant in f (x, y) and Differential (w. r to) x is written in form

$\partial f/\partial x = \partial z/\partial x = \partial F/\partial x$ or

$D_X f = Z_x = F_x$

Using same way to find $\partial f/\partial y$ is simply regards x as constant in f (x, y) and Differential (w. r. to) y is written in form

$\partial f/\partial y = \partial z/\partial y = \partial F/\partial y$ or

$D_y f = Z_y = F_y$

Since a partial derivative of function twice variables to obtain second partial derivative as

1- $\partial f/\partial x = f_x$

2- $\partial f/\partial y = f_y$

3- $\partial/\partial x\ (\partial f/\partial x) = \partial^2 f/\partial x^2 = f_{xx}$

4- $\partial/\partial y\ (\partial f/\partial y) = \partial^2 f/\partial y^2 = f_{yy}$

5- $\partial/\partial x\ (\partial f/\partial y) = \partial^2 f/\partial x\partial y = f_{yx}$

6- $\partial/\partial y\ (\partial f/\partial x) = \partial^2 f/\partial y\partial x = f_{xy}$

Note I

It is easy to extend the partial derivative of function of three variables or more

$\partial/\partial x\ (\partial^2 f/\partial y\partial x) = \partial^3 f/\partial x\partial y = f_{xyx}$

Theorem

If f (x, y) and it's partial derivatives f_x , f_y , f_{yx} ,and f_{xy} are define in region containing a point (a, b) and are all continuous at (a, b), then $f_{yx} = f_{xy}$.

Example1

Let $f(x, y) = x^2 - y^2 + xy + 7$.

Then find f_x , f_y , f_{xx} ,and f_{xy}

Solution

$f_x = 2x + y$

$f_y = -2y + x$

$f_{xy} = 1$.

Problem

1-Let $f(x, y) = e^{-x} \sin y + e^y \cos x + 8$

Then find f_x , f_y

2-Find f_x and f_y at point (1,3/2) if $f = \sqrt{4 - x^2 + y^2}$

3-If $f(x, y) = x e^y - \sin(x/y) + x^3y^2$.

Then find f_x , f_y , f_{xx} , F_{yy} and f_{xy}

4-If $U = x^2y + \arctan(xz)$, then find U_x , U_y and U_z .

5-If $V = x^2 + y^2 + z^2 + \mathrm{Log}(xz)$, then find V_x , V_y , V_z ,V_{xy} and V_{zz} .

6-If $f = x^y$, then find f_x , f_y .

9- $f = \frac{x}{}$ $x = e^t$ $y = 2t - 6$

3-

The total differential of function

$W = f(x, y, z)$,……………………………….(1)

Is defined to be

$$dw = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz$$

Or

$$dw = f_x\, dx + f_y\, dy + f_z\, dz$$

In general the total differential of function

$W = f(x, y, z, u, \ldots\ldots, v)$ is defined by

$$dw = f_x\, dx + f_y\, dy + f_z\, dz + f_u\, du + \ldots\ldots + f_v\, dv$$

where x , y, z, u…….and v are independent variables.

But if x, y and z are not independent variabl but are them can selves given by

$x = x(t),\ y = y(t),\ z = z(t)$,

then we have

$$dx = \frac{\partial x}{\partial t} dt,\ dy = \frac{\partial y}{\partial t} dt,\quad dz = \frac{\partial z}{\partial t} dt.$$

Or in the form:-

$x = x(r, s),\ y = y(r, s),\ z = z(r, s)$.

Then we ha

$$\left.\begin{aligned} dx &= \frac{\partial x}{\partial r} dr + \frac{\partial x}{\partial s} ds \\ dy &= \frac{\partial y}{\partial r} dr + \frac{\partial y}{\partial s} ds \\ dz &= \frac{\partial z}{\partial r} dr + \frac{\partial z}{\partial s} ds \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Then (1) become in case

$W = f(x, y, z) = f(x(r, s),\ y(r, s),\ z(r, s))$

$= f(r, s)$.

Then from (2) and (3) we obtain:-

$$dw = \frac{\partial w}{\partial x} dx + \frac{\partial w}{\partial y} dy + \frac{\partial w}{\partial z} dz$$

$$dw = \left[\frac{\partial x}{\partial r} dr + \frac{\partial x}{\partial s} ds\right] \frac{\partial w}{\partial x} + \left[\frac{\partial y}{\partial r} dr + \frac{\partial y}{\partial s} ds\right] \frac{\partial w}{\partial y} + \left[\frac{\partial z}{\partial r} dr + \frac{\partial z}{\partial s} ds\right] \frac{\partial w}{\partial z}$$

$=[\frac{\partial w}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial w}{\partial y}\frac{\partial y}{\partial r}+\frac{\partial w}{\partial z}\frac{\partial z}{\partial r}]\,dr+[\frac{\partial w}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial w}{\partial y}\frac{\partial y}{\partial s}+\frac{\partial w}{\partial z}\frac{\partial z}{\partial s}]ds$…….(4)

Example3

Find the total differential of function

$W = x^2 + y^2 + z^2$ if

$x = r\cos s$, $y= r\sin s$ and $z = r$

Solution

$dw = w_x\,dx + w_y\,dy + w_z\,dz$

$= 2x dx + 2y\,dy + 2z\,dz$

$dx = \frac{\partial x}{\partial r}dr+\frac{\partial x}{\partial s}ds$, or

$dx = x_r\,dr + x_s\,ds = \cos s\,dr - r\sin s\,ds$

$dy = y_r\,dr + y_s\,ds = \sin s\,dr + r\cos s\,ds$,

$dz = z_r\,dr + z_s\,ds = dr$.

Now $dw = 2x[\cos s\,dr - r\sin s\,ds] + 2y\,[\sin s\,dr + r\cos s\,ds]$
$+ 2r\,[dr.]$

$= 2\,r\cos s\,[\cos s\,dr - r\sin s\,ds] + 2\,r\sin s\,[\sin s\,dr + r\cos s\,ds] + 2r\,[dr.]$

$dw = 2[r\cos^2 s]dr - r\sin s\,ds] + 2y\,[\sin s\,dr + r\cos s\,ds]$
$+ 2r\,[dr.]$

$= 2\ [r\cos^2 s + r\sin^2 s + r]\,dr + 2\,[-r^2\sin s\cos s + r^2\sin s\cos s\,]\,ds\ + 2r\,dr.]$,

$dw = 4r\,dr]$.

Problem

If $U = f(x, y)$ Find dU, in the following:-

1- $U = 2Lnx + Lny^2$ if, $x = e^{-t}$, $y= e^t$.

2- $U = \tan^{-1}x + = \sqrt{1-y^2}$ if, $x = t^2$, $y= t-1$.

3- $U = \sin(x+ y) +\cos xy$, , $x = \pi +2t$, $y= \pi-4t$.

4- $U = x^2 + y^2+ 6xy$, $x = 3t-1$, $y= 4t-3$.

5- $U = \frac{x}{y}$, $x = \operatorname{sech} t$, $y= \coth t$.

6- $U = \tanh^{-1}(\frac{r}{s})$, $r = x\sin yz$, $s= x\cos yz$

7-$U = \ln(r+ s + t)$if, $r = xy$, $s= x z$, $t= y z$.

8- If $f(x, y) = x\cos y+y\,e^x$. Prove that $f_{yx} = f_{xy}$.

9- If $f(x, y) = \tan^{-1}(\frac{x}{y})$. Prove that $f_{xx} + f_{yy} = 0$

10- If $f(x, y) = e^{-2y}\,x\cos 2x$. Prove that $f_{xx} + f_{yy} = 0$

11- If W= sin(x+ ct). Prove that $W_{tt} = c^2 W_{xx}$.

12- If W= cos(2x+ 2ct). Prove that $W_{tt} = c^2 W_{xx}$.

13- If W= Ln(2x+ 2ct)+ cos(2x+ 2ct). Prove that $W_{tt} = c^2 W_{xx}$.

14- If W= tan(x- ct). Prove that $W_{tt} = c^2 W_{xx}$.

CHAPTER Six
Differential Equations (d.e)

Introduction:-

Definition 1.1

Differential Equations (d.e)

If y is a function of x, where y is called the dependent variable and x is called the independent variable. A differential equation is a relation between x and y which includes at least one derivative of y with respect to (w.r.to) x. Which has two types:-

1- Ordinary d.e.
   If (d.e) involves only a single independent variable this derivatives are called ordinary derivatives, and the equation is called ordinary (d.e).
2- Partial d.e.
   If there are two or more independent variables derivatives are called partial derivatives, and the equation is called partial (d.e).

For example

(a) $\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - y = 2e^x$

(b) $\partial^2u/\partial x^2 + \partial^2u/\partial y^2 = 0$

(c) $\frac{df}{dx} + x = \sin x$

(d) $y''' - 3y'' + y = 0$

The Order of (d.e)

Is that the derivative of highest order in the equation for example (a) order 3 (b) order 2 (c) order 1 (d) order 3?

Solution of Differential Equations

Any relation between the variables that occur in (d.e) that satisfies the equation is called a solution or when y and it's derivatives are replace through out by f(x) and it's derivatives for example

Show that y= acos2x +bsin2x, of derivative a solution of (d.e)

$y'' + 4y = 0$ ………………………………………………………….. (1),

Where a and b are arbitrary constant.

Solution

Since $y = a\cos 2x + b\sin 2x$,

$y' = -2a\sin 2x + 2b\cos 2x$

y'' $-4a\cos 2x - 4b\sin 2x$, put y and y'' in (1)

$-4a\cos 2x - 4b\sin 2x + 4(a\cos 2x + b\sin 2x) = 0$

0 =0, then this solution called the general solution.

Exercises

Show that each equation is a solution of the indicated (d.e)

(1) $y''' = y''$ where $y = c_1 + c_2x + c_3e^x$

(2) $x y'' + y' = 0$ where $y = c_1 \ln x + c_2$

(3) $y'' + 9y = 4\cos x$ where $2y = \cos x$

(4) $y'' - y = e^{2x}$ where $y = e^{2x}$

(5) $y'' = 2y\sec^2 x$ where $y = \tan x$.

First Order Differential Equations

The first order differential equation take in the form:-

$M(x,y)\,dx + N(x,y)\,dy = 0$……………………………………(2)

Where M and N are functions of x and y or both.

To solve this type of (d.e), we consider the following methods:-

1-**Variable Separable**

Any (d.e) can be put in the form:-

$f(x)dx + g(x)dx = 0$, or x and derivative of x in term and y derivative of y in another term.

This equation called **Variable Separable**, this equation can be solve by take the integral of two sides of this equation

$\int f(x)dx + \int g(x)dy = c$, where c is arbitrary constant.

Example

Solve xdy=ydx

ydx-xdy=0

$(ydx - xdy = 0)\,\frac{1}{xy}$,

$\frac{dx}{x} - \frac{dy}{y} = 0$, by integral of two sides

$\int \frac{dx}{x} - \int \frac{dy}{y} = c$,

Lnx-lny=c,

$\ln\frac{x}{y} = c$,

$\frac{x}{y} = e^c = c_1$,

$\therefore y = \frac{x}{c_1}$.

Problems

Solve the following differential equations:-

1- $x(2y-3)dx + (x^2+1)dy = 0$

2- $x^2(y^2+1)dx + y\sqrt{x^3+1}\,dy = 0$

3- $\frac{dy}{dx} = e^{x-y}$

4- $\sqrt{xy}\,\frac{dy}{dx} = 1$

5- $e^y \sec x dx + \cos x dy = 0$.

2-**Homogeneous Differential Equation** (H.d.e)

The differential equation as form

$M(x,y)\,dx + N(x,y)\,dy = 0$,

Where M and N are functions of x and y is called (H.d.e) if satisfy the condition

$$\left.\begin{aligned} M(kx, ky) &= k^n M(x, y) \\ N(kx, ky) &= k^n N(x, y) \end{aligned}\right\} \text{Where k is constant.}$$

For example

1- $(x^2 - y^2)dx + 2xydy = 0$

$M = x^2 - y^2$, $N = 2xy$

$M(kx,ky) = (kx)^2 - (ky)^2 = k^2x^2 - k^2y^2 = k^2(x^2 - y^2)$

$k^2(M)$

$N(kx,ky) = 2(k^2xy) = k^2(2xy)$

$k^2(N)$.

The equation is (H.d.e).

3- Solve $(x-y)dx + xydy = 0$

$M = x - y$, $N = xy$

$M(kx,ky) = (kx) - (ky) = k(x - y)$

$k(M)$

$N(kx,ky) = k^2(xy)$

$k^2(N)$.

The equation is not (H.d.e).

If the equation is homogeneous we can solve by the following method:-

Put (H.d.e) in the form

$\frac{dy}{dx} = f(y/x)$……………………………………………… (3)

Let $v = y/x$……………………………………………….. (4)

Put (4) in (3) $\Rightarrow \frac{dy}{dx} = f(v)$…………………………………..(5)

From (4) $y = xv$, and

$dy = xdv + vdx$, divided by dx

$$\frac{dy}{dx} = x\frac{dv}{dx} + v, \text{ since } \frac{dy}{dx} = f(v) \text{ from } (5)$$

$$f(v) = x\frac{dv}{dx} + v \Rightarrow f(v) - v = x\frac{dv}{dx}$$

$$(f(v)-v)dx = xdv \Rightarrow \frac{dv}{f(v)-v} = \frac{dx}{x}$$

$\frac{dx}{x} + \frac{dv}{v - f(v)} = 0$. Or

…………………………………………… (6) $\frac{dx}{x} - \frac{dv}{f(v) - v} = 0$

After solving replace v by y/x.

Example

Solve $(x^2 + y^2)\,dx + 2xydy = 0$

Solution

Since this equation (H.d.e). Now

$2xydy = -(x^2 + y^2)dx$,

$\frac{dy}{dx} = -\frac{x^2 + y^2}{2xy}$, put $y = xv$

$$\frac{dy}{dx} = -\frac{x^2 + x^2v^2}{2x(xv)} = -\frac{1+v^2}{2v}$$

$\therefore$ $f(v) = -\frac{1+v^2}{2v}$,

$$\frac{dx}{x} + \frac{dv}{v - f(v)} = 0,$$

$$\frac{dx}{x} + \frac{dv}{v + \frac{1+v^2}{2v}} = 0,$$

$\frac{dx}{x} + \frac{2vdv}{1+3v^2} = 0$, by integral both sides

$\ln x + 1/3\ln(1+3v^2) = c$,

$\ln x + 1/3\ln(1+3y^2/x^2) = c$

$$\therefore \ln x\sqrt[3]{1 + \frac{3y^2}{x^2}} = c,$$

$$x\frac{\sqrt[3]{x^2 + 3y^2}}{x} = c_1, \quad where \quad c_1 = e^c$$

$$\sqrt[3]{x^2 + 3y^2} = c_1$$

Problems

Solve the following differential equations:-

(1) $2xydx - (xy + x^2)dy = 0$

(2) $(x^2 + 2y^2 + 3xy)dx + x(x - 2y)dy = 0$

(3) $(12x^2y - 4y^3)dx + x(3y^2 - 6x^2)dy = 0$

(4) $(xe^{y/x} - ye^{y/x})dx + xe^{y/x}dy = 0$

(5) $(3x + xe^{y/x} - ye^{y/x})dx + xe^{y/x}dy = 0$

3-**Exact Differential Equation**

The differential equation as form

$M(x,y)\,dx + N(x,y)\,dy = 0$……………………………………….……..(7)

There is function

$f(x,y)=c$…………………………………………………….……...(8),

Which a solution of (7).

$df(x,y)=0$……………………………………………….………..(9).

From total partial differential equation

$df(x,y) = \partial f/\partial x\, dx + \partial f/\partial y\, dy$……………………………….(10).

From 7,8,9 and 10,

$$\left.\begin{aligned}\partial f/\partial x &= M\\ \partial f/\partial y &= N\end{aligned}\right\}\quad \ldots\ldots (11)$$

Now

$\partial^2 f/\partial y\partial x = \partial M/\partial y$, $\partial^2 f/\partial x\partial y = \partial N/\partial x$,

Since $\partial^2 f/\partial y\partial x = \partial^2 f/\partial x\partial y$

$\boldsymbol{\partial M/\partial y = \partial N/\partial x}$**,** …………………………………………………. (12).

Which condition of exact?

To solve equation (7) we must find **f** which the solution of equation (7).

$\partial f/\partial x = M$ from (11),

$\partial f = M\,\partial x$,

$f = \int M\,\partial x + A(y)$ ……………………………………………….(13),

Where A(y) is function of y.

We must find A(y)

$\partial f/\partial y = N = \partial/\partial y\,[\int M\,\partial x] + A'(y)$, since $\partial f/\partial y = N$, from (11),

$\therefore A'(y) = N - \partial/\partial y\,[\int M\,\partial x]$

$\therefore A(y) = \int\{N - \partial/\partial y\,[\int M\,\partial x]\} + c$………………………………..(14)

Now put (14) in (13) which complete solution.

Example

Solve the following differential equation

$$\frac{dy}{dx} = \frac{xy^2 - 1}{1 - x^2 y}$$

Solution

$(1 - x^2y)dy - (xy^2 - 1)dx = 0$,

$N = 1 - x^2y$, $M = 1 - xy^2$,

$\partial M/\partial y = \partial N/\partial x = -2xy$,

$\partial f/\partial x = M$,

$f = \int M\,\partial x + A(y) = \int (1 - xy^2)\,\partial x + A(y)$

$f = (x - (x^2 y^2)/2 + A(y)$…………….(*)

(we must find A(y),

$\partial f/\partial y = -x^2 y + A'(y) = N = 1 - x^2 y$

$\therefore\ A'(y) = 1$,

$A(y) = y + c$ put in (*)

$F = (x - (x^2 y^2)/2 + y + c$.

Problems

Solve the following differential equations:-

(1) $(2x + y)dx + (x + y)dy = 0$

(2) $(3x - y)dx - (x - y)dy = 0$

(3) $(\cos x + y)dx + (2y + x)dy = 0$

(4) $(ye^x + y)dx + (x + e^x)dy = 0$

(5) $\tan y\, dx + x \sec^2 y\, dy = 0$.

Integrating Factor

So

M

M

4- First – Order Linear Differential Equation

If the equation as form:-

$\frac{dy}{dx}$ +P(x) y = Q(x)…………………………………………… (15)

Where P and Q are functions of x.

To solve equation (15), we must find (I) where

I= $e^{\int pdx}$ { I is integrating factor}.

Now multiple both sides of (15) by I

dy + Py dx = Q dx………………………………………………(15)

$e^{\int pdx}$ { dy + Py dx = Q dx }

$e^{\int pdx}$ dy + $e^{\int pdx}$ Py dx = $e^{\int pdx}$ Q dx }

d [y$e^{\int pdx}$]=Q $e^{\int pdx}$ dx……………………………………………(16)

by integrate (16)

y$e^{\int pdx}$ = ∫Q $e^{\int pdx}$ dx +c,

Which the solution of (15), or the solution is

Iy = ∫IQ dx +c

Example

Solve the following differential equation

$\frac{dy}{dx} + \frac{y}{x} = 2$

Sol

Since P =1/x, Q = 2

I= $e^{\int pdx}$ = I= $e^{\int dx/x}$ = $e^{\ln x}$ = x.

The solution

Iy = ∫IQ dx +c

xy = ∫2x dx +c,

xy = x^2 +c

y =x + c/x

Problems

Solve the following differential equations:-

(1) y′ + 2y= e^x

(2) xy′ +3y =x^2

(3) y′ + ycotx = cosx

(4) x y′ +2y = x^2-x +1

(5) y′ -y tanx =1.

4.1 The Bernoulli Equation

The equation

$\frac{dy}{dx}$ +P(x) y = Q(x) y^n ……… (*), if n≠ 1.

Is similar to L-equation is called Bernoulli Equation.

We shall show how transform this equation to linear equation. In fact we must reduce this equation to linear, product (*) by (y^{-n}) or

$[\frac{dy}{dx} + P(x)\, y = Q(x)\, y^n]\, y^{-n}$

$\frac{dy}{dx} y^{-n} + P y^{1-n} = Q$ …………………. (**).

Let $w = y^{1-n} \Rightarrow dw = (1-n)\, y^{-n}\, dy$ or

$\frac{dw}{1-n} = y^{-n}\, dy$ put in (**)

$$\left.\begin{aligned} &\frac{dw}{(1-n)dx} + P y^{1-n} = Q, \\ &\quad \text{or} \\ &\frac{dw}{dx} + (1-n)P\, w = (1-n)\, Q \end{aligned}\right\} \text{ Which is L-equation}$$

Example

Solve the following differential equation

$\frac{dy}{dx} + \frac{y}{x} = y^2$

Sol

$[\frac{dy}{dx} + \frac{y}{x} = y^2]\, y^{-2}$

$\frac{dy}{dx} y^{-2} + \frac{y^{-1}}{x} = 1$…………. (#),

Let $w = y^{-1} \Rightarrow dw = - y^{-2}\, dy$

$-dw = y^{-2}\, dy$, put in (#)

$- \frac{dw}{dx} + \frac{w}{x} = 1$

$\frac{dw}{dx} - \frac{w}{x} = -1.$

$P = - \frac{1}{x}$, $Q = - 1$,

$I = e^{\int p dx} = I = e^{-\int dx/x} = e^{-\ln x} = \frac{1}{x}.$

The solution

$Iw = \int IQ\, dx + c,$

$\frac{w}{x} = \int - \frac{1}{x}\, dx + c,$

$$\frac{w}{x} = -\ln x + c,. \text{ Since } w = y^{-1} = \frac{1}{y}$$

$$\frac{1}{xy} = -\ln x + c,$$

$$y = \frac{1}{x(c - \ln x)}.$$

Problems

Solve the following differential equations:-

(1) $y' - 2y/x = 4x y^2$
(2) $y' + y/x = x y^2$
(3) $y' + y = y^3 e^{2x} \sin x$
(4) $y' + 2y/x = 5 y^2/x^2$
(5) $x y' + y = y^2$.

Second – Order Differential Equation

Special Types

Certain types of second order differential equation such that

$$F(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}) \ldots\ldots(17)$$

Can be reduced to first order equations by a suitable of variables:-

Type I

Equation with dependent variable when equation as form

$$F(x, \frac{dy}{dx}, \frac{d^2y}{dx^2}) \ldots\ldots(18)$$

It can be reduced to first order equation by suppose that:-

$$p = \frac{dy}{dx}, \qquad \frac{d^2y}{dx^2} = \frac{dp}{dx}$$

Then equation (18) takes the form

$$F(x, p, \frac{dp}{dx}) = 0,$$

Which is of the first order in p, if this can be solved for p as function of x says?

$p = q(x,c_1)$.

Then y can be found from one additional integration

$$y = \int (\frac{dy}{dx})\, dx + c = \int p\, dx + c = \int q(x,c_1)\, dx + c.$$

Type II

Equation with independent variable when equation (17) does not contain x explicit but has the form

$F(y, \frac{dy}{dx}, \frac{d^2y}{dx^2})=0$……………………………………………..…….(19)

The substitution to use are :-

$p = \frac{dy}{dx}$, $\frac{d^2y}{dx^2} = \frac{dp}{dx} = \frac{dp}{dy} \cdot \frac{dy}{dx} = \frac{dp}{dy} p$

The equation (19) become

$F(y, p, p\frac{dp}{dx}) = 0$,

Which is of the first order in p. Which solution gives p in terms of y, and then further integration gives the solution of equation (19).

Example 1

Solve the following differential equation

$\frac{d^2y}{dx^2} + \frac{dy}{dx} = 0$…………………………………..……………. (*)

Let $p = \frac{dy}{dx}$, $\frac{d^2y}{dx^2} = \frac{dp}{dy} p$, put in (*)

$\frac{dp}{dy} p + p = 0$, $\div p$

$\frac{dp}{dy} + 1 = 0$,

dp +dy =o. by integration

$p + y = c_1$

$\frac{dy}{dx} + y = c_1$

$\frac{dy}{dx} = c_1 - y$

$\frac{dy}{c_1 - y} = dx \Rightarrow -\ln(c_1 - y) = x + c_2$

$\ln(c_1 - y) = -x + c_2 \Rightarrow c_1 - y = e^{-x+c_1} = ce^{-x}$

$y = c_1 - ce^{-x}$

Example 2

Solve the following differential equation

$X^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} = 1$

$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} = 1/x^2$……………………………………………. (**)

Let $p = \frac{dy}{dx}$ and $\frac{d^2y}{dx^2} = \frac{dp}{dx}$, put in (**)

$\frac{dp}{dx} + \frac{1}{x} p = 1/x^2$, which linear in p,

$I = e^{\int p dx} = I = e^{\int dx/x} = x.$

$Ip = \int IQ\ dx + c \Rightarrow Ip = \int x(1/x^2)\, dx + c = \ln x + c$

$\therefore\ xp = \ln x + c$

$P = (\ln x)/x + c/x$

Let $\frac{dy}{dx} = (\ln x)/x + c/x$

$dy = [(\ln x)/x]dx + (c/x)dx,$

$y = (\ln x)^2/2 + c \ln x + c_1$

Problems

Solve the following differential equations:-

(1) $y'' + y' = 0$

(2) $y'' + y y' = 0$

(3) $\mathbf{x} y'' + y' = 0$

(4) $y'' - y' = 0$

(5) $y'' + w^2 y = 0$, where w constant $\neq 0$.

Homogeneous-Second – Order (D. E) With Constant Coefficient

Consider linear equation with constant coefficient which in the form:-

$y'' + a y' + by = 0$…………………………………………………… (20)

where a, b are constant.

How to solve this equation we shall now find how to determine m such that

$y = e^{mx}$ is a solution of (20) then

$y' = me^{mx}$ and $y'' = m^2 e^{mx}$, put in (20)

$m^2 e^{mx} + a\, me^{mx} + be^{mx} = 0$

$e^{mx}(m^2 + a\, m + b) = 0,$

since $e^{mx} \neq 0$, then

$m^2 + a\, m + b = 0$…………………………………………………. (21).

Which called **characteristic equation**.

Then we saw that e^{mx} is a solution of (20) $\Leftrightarrow$ m is root of (21).

Note

The general solution of (20), there is three cases:-

Case i

If $m_1 = m_2$ in equation (21), the solution of (20) (homogeneous equation) is

$y_h = (c_1 + xc_2)\, e^{mx}$

Case ii

If $m_1 \neq m_2$ in equation (21), the solution of (20) (homogeneous equation) is

$y_h = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

Case iii

If m_1 and m_2 roots (m= $\alpha + i\beta$ where i= $\sqrt{-1}$) in equation (21), the solution of (20) (homogeneous equation) is

$y_h = e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x)$

Ex i

Solve $y'' +4y' +4y=0$……………………………………………..(*)

Sol

let $y= e^{mx}$, $y' = me^{mx}$ and $y'' =m^2e^{mx}$, put in (*)

$m^2e^{mx} +4\, me^{mx} +4e^{mx} =0$

$e^{mx} (m^2 +4\, m+4) =0$,

since $e^{mx} \neq 0$, then

$m^2 +4\, m+4 =0$

Which called **characteristic equation?**

$(m+2)^2 =0 \Rightarrow m_1 =m_2 = -2$, the solution of (*) is

$y_h= (c_1 + xc_2)\, e^{-2x}$

Ex ii

Solve $y'' +y' -6y=0$…………………………………………..(**)

Sol

let $y= e^{mx}$, $y' = me^{mx}$ and $y'' =m^2e^{mx}$, put in (*)

$m^2e^{mx} + me^{mx} -6e^{mx} =0$

$e^{mx} (m^2 + m-6) =0$,

since $e^{mx} \neq 0$, then

$m^2 +m-6 =0$

This called characteristic equation

$(m+3)(m-2) =0 \Rightarrow$ either $m_1 = -3$ or $m_2 =2$, the solution of (**) is

$y_h= c_1 e^{-3x} + c_2 e^{2x}$

Ex iii

Solve $y'' -4y' +5y=0$…………………………………………..(**)

Sol

let $y= e^{mx}$, $y' = me^{mx}$ and $y'' =m^2e^{mx}$, put in (*)

$m^2e^{mx} -4me^{mx} +5e^{mx} =0$

$e^{mx} (m^2 -4\, m+5) =0$,

since $e^{mx} \neq 0$, then

$m^2 -4m+5 =0$

This called characteristic equation

$m_1 = \frac{4 \pm \sqrt{16-20}}{2} = \frac{4 \pm \sqrt{-4}}{2}$,

$m_2 = 2 \pm i \Rightarrow \alpha + i\beta$, $\alpha =2$, $\beta =1$,

$y_h= e^{2x} (c_1 \cos x + c_2 \sin x)$.

<u>Non-Homogeneous-Second – Order (D. E) With Constant Coefficient</u>

Consider the equation which in the form:-

$y'' + a y' + by = f(x)$……………………………………………… (22)

where a, b are constant.

To find the general solution of (22). We find solution of homogeneous part

$y'' + a y' + by = 0$………………………………………………… (23),

let y_h be solution of (23).

Then the solution of (22) take by added the solution y_h to any another special solution y_p of (22)such that the general solution of (22) become

$y(x) = y_h + y_p$

Method of Undetermined Coefficient

Variation of Parameter

Consider the equation which in the form:-

$y'' + a y' + by = f(x)$ …………………………………………… (24)

Where a, b are constant, f(x) be any function of x.

To solve (24)

(a) Find y_h (solution of (H-part),

$y_h = c_1 u_1 + c_2 u_2$ ……………………………………………(25)

Where c_1 and c_2 are arbitrary constant, and u_1 and u_2 are two function as form:-

let e^{mx} or xe^{mx} $e^{\alpha x}\cos\beta x$ or $e^{\alpha x}\sin\beta x$), which solution of (H-part).

(b) We replace c_1 and c_2 by function of x say v_1 and v_2 then (#) become

$y_h = c_1 v_1 + c_2 v_2$ ……………………………………………..(26),

Which solution of (24),

$y_h' = v_1u'_1 \; v'_1 u_1 + v_2 u'_2 + v'_2 u_2$

$y_h' = (v_1u'_1 + v_2 u'_2) + (v'_1 u_1 + + v'_2 u_2)$ ……………………..…(27),

from this

$y_h' = v_1u'_1 + v_2 u'_2$,

and

$v'_1 u_1 + + v'_2 u_2 = 0$ ………………………………………………..(28)

Now

$y_h'' = v'_1 u'_1 + v_1u''_1 + v'_2 u'_2 + v_2 u_2''$

(c) Now put y_h, y_h' and y_h'' in (24)

$v'_1 u'_1 + v_1u''_1 + v'_2 u'_2 + v_2 u_2'' + a[v_1u'_1 + v_2 u'_2] + b[c_1 v_1 + c_2 v_2] = f(x)$

$v_1 [u_1'' + a u_1' + b u_1] + v_2[u_2'' + a u_2' + b u_2] + v'_1 u'_1 + v'_2 u'_2 = f(x)$

Since $y'' + a y' + by = 0$,

$\therefore u_1'' + a u_1' + b u_1 = o$, $u_2'' + a u_2' + b u_2 = o$, and

$v'_1 u'_1 + v'_2 u'_2 = f(x)$.

{the value in brackets vanish because by hypothesis both u_1 and u_2 are solution of homogeneous equation corresponding to (24).

Then the equation (24) satisfy by equation

$\mathbf{v'_1 u_1 + v'_2 u_2 = 0}$ ……………………………….................... (29)

$\mathbf{v'_1 u'_1 + v'_2 u'_2 = f(x)}$ ………………………………………… (30).

(d) By solve (29) and (30) we find two unknown $\mathbf{v'_1, v'_2}$ and we find $\mathbf{v_1}$, $\mathbf{v_2}$ by integral.

(e) The general solution of (non- H. D.E) (24) is

$\mathbf{Y(x) = v_1 u_1 + v_2 u_2}$

Example1

Find the general solution of the following (d.e)

$y'' - y' - 2y = e^{-x}$ ………………………………………………… (*)

Sol

(1) Find the general solution of (H.d.e) $y'' - y' - 2y = 0$, or

$m^2 - m - 2 = 0$,

$(m-2)(m+1) = 0 \Rightarrow$

$y_h = c_1 e^{2x} + c_2 e^{-x}$,

$u_1 = e^{2x}$, $u_2 = e^{-x}$,

$u'_1 = 2e^{2x}$, $u'_2 = -e^{-x}$

$\mathbf{v'_1 u_1 + v'_2 u_2 = 0}$……………………#

$\mathbf{v'_1 u'_1 + v'_2 u'_2 = f(x)}$………………. ##

$v'_1 e^{2x} + v'_2 e^{2x} = 0$……………………#

$2v'_1 e^{2x} - v'_2 e^{-x} = e^{-x})$………………. ##

$v'_1 e^{2x} = e^{-x} \Rightarrow v'_1 = 1/3e^{-3x} \Rightarrow v_1 = -1/9e^{-3x} + c_1$,

From (#)$v'_1 e^{2x} = - v'_2 e^{-x}$, or $v'_2 =- v'_1 e^{3x}$

$v'_2 = -1/3 \Rightarrow$

$v_2 = -1/3x + c_2$

Since $\mathbf{Y(x) = v_1 u_1 + v_2 u_2}$

$\mathbf{Y(x)} = (-1/9e^{-3x} + c_1)e^{2x} + (-1/3x + c_2)e^2$.

Problems

Solve the following differential equations:-

(1) $y'' + 4y' = 3x$

(2) $y'' - 4y' = 8x^2$

(3) $y'' - y' - 2y = 10\cos x$

(4) $y'' - 4y' + 3y = e^x$

(5) $y'' + y = \sec x$.

Problems

Solve the following differential equations:-

1- $x(2y-3)dx + (x^2 + 1)dy = 0$

2- $x^2 (y^2+1)\, dx + y\sqrt{x^3 + 1}\, dy = 0$

3- $\sin x \frac{dx}{dy} + \cosh 2y = 0$

4 - $\sqrt{2xy}\frac{dy}{dx} = 1$

5- $\ln x \frac{dx}{dy} = \frac{x}{y}$

6- ($xe^y\, dy + \frac{x^2+1}{y}\, dx = 0$

7- $y\sqrt{1+x^2}\, dy + \sqrt{y^2 - 1}\, dx = 0$

8- $x^2y \frac{dy}{dx} = (1+x) \csc y$

9- $\frac{dy}{dx} = e^{x-y}$

10- $e^y \sec x\, dx + \cos x\, dy = 0$

(H. d. e)

11- $(x^2 + y^2)\, dx + xy dy = 0$

12 - $x^2\, dx + (y^2 - xy)dy = 0$

13 – $x\, e^{y/x} + y)dx - xdy = 0$

14 –$(x + y)\, dy + (x - y)\, dx = 0$

15 - $\frac{dy}{dx} = \frac{y}{x} + \cos\left(\frac{y-x}{x}\right)$

16- $xdy - 2ydx = 0$

17- $2xydy + (x^2 - y^2)dx = 0$

(Linear d. e)

18- $\frac{dy}{dx} + 2y = e^{-x}$

19- $x\, y' + 3y = \frac{\sin x}{x^2}$

20- $2\, y' - y = e^{x/2}$

21- $xdy + ydx = \sin x\, dx$

22- $xdy + ydx = ydy$

23- $(x-1)^3\, y' + 4\, (x-1)^2\, y = x+1$

24- $\text{Cosh}x\, dy + (y \sinh x + e^x)\, dx = 0$

25- $e^{2y}\, dx + 2(x\, e^{2y} - y)dy = 0$

26 – $(x-2y)\, dy + y\, dx = 0$

27 – $(y^2 + 1)\, dx + (2xy + 1)\, dy = 0$

(Exact d. e)

Use the given integrating factor to make (d. e) exact then solve the equation

28 - $(x+2y)\, dx - x\, dy = 0$, $\quad (I = \frac{1}{x^3})$

29 – $y\, dx + x\, dy = 0$, $\quad (I = \frac{1}{xy})$ or $(I = \frac{1}{(xy)^2})$

Solve (exact d. e)

30 - $(x + y)\, dx + (x+y^2)\, dy = 0$

31 – $(2xe^y + e^x)\, dx + (x^2 + 1)\, e^y\, dy = 0$

32 – $(2xy + y^2)dx + (x^2 + 2xy - y)dy = 0$

33- $(x+\sqrt{y^2+1})dx - (y- \dfrac{xy}{\sqrt{y^2+1}})dy = 0$

34- $x\,dy + y\,dx + x^3 dx = 0$

35- $x\,dy - y\,dx = x^2\,dx$

36- $(x^2 + x - y)\,dx + x\,dy = 0$

37- $(e^x + \ln y + \dfrac{y}{x})\,dx + (\dfrac{x}{y} + \ln x + \sin y)\,dy = 0$

38- $(\dfrac{y^2}{1+x^2} - 2y)\,dx + (2y \tan^{-1}x - 2x + \sinh y)dy = 0$

39- $dy + \dfrac{y - \sin x}{x}\,dx = 0$

(Second- Order)

40- $y'' + 2y = 0$

41- $y'' + 5y' + 6y = 0$

42- $y'' + 6y' + 5y = 0$

43- $y'' - 6y' + 10y = 0$

44- $y'' + y = 0$

45- $y'' + y' \quad x$

46- $y'' + y = \sin x$

47- $y'' - 2y' + y = e^{-x}$

48- $y'' + 2y' + y = e^x$

49- $y'' - y = \sin x$

50- $y'' + 4y' + 5y = x + 2$.

51- $y'' - y = e^x$

52- $y'' + y = \sec x$

53- $y'' + y = \tan x$

54- $y'' + y = \cot x$

CHAPTER Seven

Laplace Transformation (L. T)

Definition 2.1

Let f (t) be function of variable t which define on all value of t such that (t >0).

The Laplace transformation of f (t) which written as L {f (t)} is

$$F(s) = L\,\{f(t)\} = \int_0^{\infty} e^{-st} f(t)\,dt \qquad (1)$$

Note 1

The Laplace transformation is define in (1) is converge to value of s, and no define if the integral in (1) has no value of s.

Laplace Transformation of Some Function:-

Using the definition (1) to obtain the following transforms:-

1- If f(t) =1

Solution

Since $L\,\{f(t)\} = \int_0^{\infty} e^{-st} f(t)\,dt = -\frac{1}{s} e^{-st} \Big|_0^{\infty} = 0 + \frac{1}{s} e^0 = \frac{1}{s}$

$\therefore L(1) = \frac{1}{s}$.

2-If f (t) = e^{at}

Solution

Since $L\,\{f(t)\} = \int_0^{\infty} e^{-st} f(t)\,dt$

$$L\,\{f(t)\} = \int_0^{\infty} e^{-st} e^{at}\,dt = L\,\{f(t)\} = \int_0^{\infty} e^{(a-s)t}\,dt$$

$$= \int_0^{\infty} e^{-(s-a)t}\,dt = -\frac{1}{s-a} e^{-(s-a)t} \Big|_0^{\infty}$$

$$= \frac{1}{s-a}$$

$$\therefore l\{e^{at}\} = \frac{1}{s-a} .$$

Note 2

Let f(t) be function and c constant then

(i) $L\,\{cf(t)\} = cL\,\{f(t)\}$

(ii) $L\,\{f_1(t) \pm f_2(t)\} = L\,\{f_1(t)\} \pm L\,\{f_2(t)\}$

3-If f(t) = cos(wt)

4-If f(t) = sin(wt).

Solution

From Euler formula

e^{-iwt} = cos (wt) + isin (wt).

L{ e^{-iwt} } = L{ cos (wt)} + i L{sin (wt)}……………………(*).

But

$l\{e^{iwt}\} = \dfrac{1}{s-iw}$, from (2)

$$\frac{1}{s-iw} = \frac{1}{s-iw} \times \frac{s+iw}{s+iw} = \frac{s+iw}{s^2+w^2}$$

$$= \frac{s}{s^2+w^2} + i \frac{w}{s^2+w^2}$$

$\therefore\ l\{e^{iwt}\} = \dfrac{s}{s^2+w^2} + i\dfrac{w}{s^2+w^2}$, from (*)

L{ cos (wt)} + i L{sin (wt)}= $l\{e^{iwt}\} = \dfrac{s}{s^2+w^2} + i\dfrac{w}{s^2+w^2}$

From this

3- L {cos (wt)} = $\dfrac{s}{s^2+w^2}$

4- L {sin (wt)} = $\dfrac{w}{s^2+w^2}$

5-If f(t) = sinh (wt).

6-If f(t) = cosh (wt).

Solution

Since sinhx = $\dfrac{1}{2}[e^x - e^{-x}]$, coshx = $\dfrac{1}{2}[e^x + e^{-x}]$.

Now sinh (wt) = $\dfrac{1}{2}[e^{wt} - e^{-wt}]$,

L {sinh (wt)} = $\dfrac{1}{2}\{L(e^{wt}) - L(e^{-wt})\}$,

$$= \frac{1}{2}\left\{ \frac{1}{s-w} - \frac{1}{s+w} \right\}.$$

$$= \frac{1}{2}\frac{2w}{s^2-w^2} = \frac{w}{s^2-w^2}$$

$\therefore$ L {sinh (wt)} = $\dfrac{w}{s^2-w^2}$, and

$L\{\cosh(wt)\} = \dfrac{s}{s^2 - w^2}$.

Example 1

Find $L\{8-6e^{3t} + e^{-4t} + 5\sin 3t + 7\cosh 3t\}$

Solution

$L(8) = 8L(1) = 8\,\dfrac{1}{s} = \dfrac{8}{s}$.

$L(6e^{3t}) = 6L(e^{3t}) = \dfrac{6}{s-3}$

$L(e^{-4t}) = \dfrac{1}{s+4}$

$L\{5\sin(3t)\} = 5L\{\sin(3t)\} = 5\,\dfrac{3}{s^2+9} = \dfrac{15}{s^2+9}$

$L\{7\cosh(3t)\} = 7L\{\cosh(3t)\} = \dfrac{7s}{s^2-9}$

Laplace Transformation of Differential

Theorem :-

If f(t)is continuous function of exponential on $[o, \infty)$ whose derivative is also exponential then the (L.T) of $f'(t)$ is given by formula

$$L\{f'(t)\} = \int_0^\infty e^{-st} f'(t)\,dt$$

Proof

$\int udv = uv - \int vdu$.

Let $u = e^{-st} \Rightarrow du = -s\,e^{-st}\,dt$,

$dv = f'(t)\,dt \Rightarrow v = f(t)$,

$$\Rightarrow \int_0^\infty e^{-st} f'(t)\,dt = e^{-st} f(t) \Big|_0^\infty + \int_0^\infty s\,e^{-st} f(t)\,dt$$

$$= o - e^0 f(0) + s\int_0^\infty e^{-st} f(t)\,dt$$

$= -f(0) + s\,L\{f(t)\}$. Where $s\int_0^\infty e^{-st} f(t)\,dt = s\,L\{f(t)\}$.

$\therefore\ L\{f'(t)\} = s\,L\{f(t)\} - f(0)$.

Corollary

If both f(t) and $f'(t)$ are continuous functions of exponential order on $[o, \infty)$, and if $f''(t)$ is also exponential then :-

$L\{f''(t)\} = s^2 L\{f(t)\} - s f(0) - f'(0)$

Proof

$$L\{f''(t)\} = L\{f'(t)\}' = sL\{f'(t)\} - f'(0)$$
$$= s[s L\{f(t)\} - f(0)] - f'(0)$$
$$= s^2 L\{f(t)\} - s f(0) - f'(0).$$

Now in general

$L\{f^n(t)\} = s^n L\{f(t)\} - s^{n-1} f(0)$ ----------- $f^{n-1}(0)$.

Problem

Prove that

$L\{t^n\} = \frac{n!}{s^{n+1}}$, where n=1, 2, 3, ……….., and n!= n(n-1)(n-2)…..(n-n).

And 0! =1.

Properties of L. T

(1) **Shifting**

If $L\{f(t)\} = f(s) = L\{e^{at} f(t)\}\ f(s-a)$

Example

Find $L\{e^{-4t}\cos 3t\}$

Solution

$f(t) = \cos 3t$, $a=-4$, then $f(s) = L\{f(t)\} = L\{\cos 3t\} = \frac{s}{s^2+9}$

$$L\{e^{-4t}\cos 3t\} = \frac{s-(-4)}{(s-(-4))^2+9} = \frac{s+4}{(s+4)^2+9}$$

(2) **L. T of Integrals**:

$$L\{\int_0^t f(u)\,du\} = \frac{f(s)}{s}$$

Example

Find $L\{\int_0^t \sinh 2t\,dt\}$

Solution

$$F(u) = L\{\sinh 2t\} = \frac{2}{s^2-4} \quad , L\{\int_0^t \text{sinsh}2t\,dt\} = \frac{\frac{2}{s^2-4}}{s}$$

$$= \frac{2}{s(s^2-4)} \quad .$$

(3) **Multiplication by t^n**

If L {f (t)} = f(s), then

$$L\{t^n f(t)\} = (-1)^n \frac{d^n f(s)}{ds^n}$$

Example

Evaluate $L\{t^2 e^{3t}\}$

Solution

$f(t) = e^{3t}$

$$L(f(t)) = L(e^{3t}) = \frac{1}{s-3} \quad = f(s)$$

$$f'(s) = \partial f/\partial s = \frac{-1}{(s-3)^2}$$

$$f''(s) = \partial^2 f/\partial s^2 = \frac{2}{(s-3)^3}$$

$$L\{t^2 e^{3t}\} = (-1)^2 \frac{2}{(s-3)^3} = \frac{2}{(s-3)^3}$$

If

L

Ex

Ev

L

t

f(t

$$= \frac{1}{s} + \frac{e^{-s}}{s} - 3\frac{e^{-2s}}{s} + \frac{e^{-3s}}{s}.$$

L. T of Periodic Functions

If f (t) is Periodic function of period T>0 satisfy such that f(x+T) = f(x), then

$$L\{f(t)\} = \frac{\int_0^T e^{-st} f(t)dt}{1-e^{-sT}}$$

Gamma Function

Definition 2.4

If (n >0), then the gamma (n) becomes:-

$$\Gamma(n)=\int_0^\infty t^{n-1}e^{-t}\, dt \qquad \ldots\ldots( )$$

Important Properties of gamma function

(i) $\Gamma(n+1) = n\,\Gamma(n)$

ii) $\Gamma(n+1) = n!$

(iii) $\Gamma(\frac{1}{2}) = \Gamma(\Pi)$

Table I

Some elementary function f (t) their Laplace Transforms L {f (t)} = f(s).

	$f(t)$	$L\{f(t)\} = f(s)$
1	1	$\frac{1}{s}$
2	t	$\frac{1}{s^2}$
3	t^2	$\frac{2!}{s^3}$
4	$t^n, n = 1,2,3,...$	$\frac{n!}{s^{n+1}}$
5	e^{at}	$\frac{1}{s-a}$
6	$\cos wt$	$\frac{s}{s^2+w^2}$
7	$\sin wt$	$\frac{w}{s^2+w^2}$
8	$\cosh at$	$\frac{s}{s^2-a^2}$
9	$\sinh at$	$\frac{a}{s^2-a^2}$
10	$y'(t)$	$sL\{y(t)\} - y(0) = sf(s) - y(0)$
11	$y''(t)$	$s^2L\{y(t)\} - sy(0) - y'(0) = s^2 f(s) - sy(0) - y'(0)$
12	$\int_0^t f(u)du$	$\frac{f(s)}{s}$
13	$t^n f(t)$	$\frac{(-1)^n \partial^n f(s)}{\partial s^n}$
14	$t^n (npositive)$	$\frac{\Gamma(n+1)}{s^{n+1}}$

Problems

Find Laplace transform (f(s)) of the following functions :-

1- $f(t) = \sin^2 t$
2- $f(t) = t^4 e^{3t}$
3- $f(t) = e^{-t} \cosh 3t$
4- $f(t) = \dfrac{\sinh t}{t}$
5- $f(t) = t^2 e^{3t}$
6- $f(t) = 3t+4$
7- $f(t) = t^2 + at + b$
8- $f(t) = (a+bt)^2$
9- $f(t) = t e^{-t}$
10- $f(t) = (e^{2t} - 4)^2$
11- $f(t) = t e^{at} \sin at$
12- $f(t) = \cosh at \cos at$
13- $f(t) = \begin{cases} 0 & \text{when } 0<t<2 \\ 4 & \text{when } 2<t. \end{cases}$

14 -Prove that $\int_0^\infty t\, e^{-3t} \sin t\, dt = \dfrac{3}{50}$

15- Prove that

(a) $= L\{a+bt\} = \dfrac{as+b}{s^2}$

(b) $= L\{t \cos at\} = \dfrac{s^2 - a^2}{(s^2 + a^2)^2}$,

Inverse Laplace Transformation

If $L\{f(t)\} = f(s)$. Then we call f(t) is the inverse of (L. T) of function f(s) and which written as:

$f(t) = L^{-1}\{f(s)\}$, for example

$L(e^{3t}) = \dfrac{1}{s-3} = f(s)$.

$L^{-1}\{f(s)\} = L^{-1}\{\dfrac{1}{s-3}\} = e^{3t}$.

Some Properties of Inverse L. T

We see the L. T of first (9) in table I, we can inverse there Laplace to find inverse of this for example

L(1) = f(s)= $\frac{1}{s}$,

L^{-1} {f(s)} = L^{-1} { $\frac{1}{s}$ } =1,

Example1

Find f (t), if f(s) = $\frac{5}{s+3}$

Solution

$$f(t) = L^{-1}\left\{\frac{5}{s-3}\right\} = 5L^{-1}\left\{\frac{1}{s+3}\right\} = 5\,e^{-3t}$$

Example1

Find f (t), if f(s) = $\frac{s+1}{s^2+1}$

Solution

$$f(s) = \frac{s}{s^2+1} + \frac{1}{s^2+1},$$

$$f(t) = L^{-1}\left\{\frac{s}{s^2+1}\right\} + L^{-1}\left\{\frac{1}{s^2+1}\right\},$$

= cost + sint.

Partial Fraction

If we want to find the inverse transform of a rational function as $\frac{f(x)}{g(x)}$, where f and g are polynomials which the degree of f less than degree of g then.

We can take advantage of partial transform is easily found as see in examples:-

Example1

Find the inverse Laplace transform (f (t)), if f(s) = $\frac{1}{s^2(s^2+1)}$

Solution

$$f(s) = \frac{1}{s^2(s^2+1)},$$

$$\frac{1}{s^2(s^2+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+1},$$

$$1 = As^3 + Bs^2 + As + B + Cs^3 + Ds^2,$$

B =1,

B+D =0,

A+ C =0,

D = -1 $\longrightarrow$ A=0 $\longrightarrow$ C=0,

$$\frac{1}{s^2(s^2+1)} = \frac{1}{s^2} - \frac{1}{s^2+1},$$

$f(t) = L^{-1}\{ \frac{1}{s^2(s^2+1)} \} = L^{-1}\{ \frac{1}{s^2} \}$ - $L^{-1}\{ \frac{1}{s^2+1} \}$,

$= t^2 - \sin t.$

Example2

Find the inverse Laplace transform (f (t)), if $f(s) = \frac{s+1}{s^2+s-6}$

Solution

$f(s) = \frac{s+1}{s^2+s-6} = \frac{s+1}{(s+3)(s--2)} = \frac{A}{s+3} + \frac{B}{s-2}.$

S+1 =As - 2A + Bs + 3B,

-2A +3B = 1,

A+B =1,

-2A +3B = 1

2A + 2B = 2 +

5B=3 ⟶ B =3/5, A = 2/5,

$f(t) = L^{-1}\{ F(s)\} = 2/5 L^{-1}\{ \frac{1}{s+3} \}$ - $3/5 L^{-1}\{ \frac{1}{s-2} \}$,

$= 2/5\ e^{-3t} + 3/5\ e^{2t}$

Problems

Find f(t) { the inverse Laplace transform} of the following:-

(1) $f(s) = \frac{1}{s^2-1}$,

(2) $f(s) = \frac{1}{s^2(s^2+1)}$,

(3) $f(s) = \frac{s^2-6}{s^3+4s^2+3s}$,

(4) $f(s) = \frac{1}{s\,(s^2+4)}$,

(5) $f(s) = \frac{4}{s-2} - \frac{3s}{s^2+16} + \frac{5}{s^2+4}$,

(6) $f(s) = \frac{1}{s+3}$,

(7) f(s) = $\frac{1}{s^2+9}$,

(8) f(s) = $\frac{2s+3}{s^2+9}$,

(9) f(s) = $\frac{s+3}{s^2+s-6}$.

Application of Laplace Transformation

Linear (D. E) With Constant Coefficient

To solve L- non homogeneous (d. e) of order n with constant coefficient.

We use same way as second- order (d. e) as form :-

$a_0 y'' + a_1 y' + a_2 y = f(x)$…………………………………… … (*)

Where a_0, a_1 and a_2 are constant, which satisfy initial condition:

y(0)= A and y′ (0) =B……………………………………………(**)

Where A and B are choice constant.

Example1

Find the solution of the following (d.e) by (L. T)

$y'' + 3y' + 2y = 0$……………………………………………… (*)

Which satisfies initial condition?

y (0)= 0 and y′ (0) =1.

Solution

L {y ″ (t)} = s^2 {y (s)} – s y (0) - y ′ (0)
L {y ′ (t)} = s {y (s)} – y (0).
L {y (t)} = y (s)
Since L {y} = y (s)

} Put in (*)

s^2 {y (s)} – s y (0) - y ′ (0) +3[s {y (s)} – y (0)] + 2 y (s) =0

By use y (0) = y′ (0) =1,

(s^2 + 3s +2) y (s) = (s+3) y (0) + y ′ (0),

(s^2 + 3s +2) y (s) = s+3+1 = s+4,

$$y(s) = \frac{s+4}{s^2+3s+2} = \frac{s+4}{(s+1)(s+2)} = \frac{A}{s+2} + \frac{B}{s+1}.$$

S+4 = As+ 2B + As + A,

A + B = 1

A +2B = 4 -

-B =-3 ⟶ B =3 ⟶ A =-2,

$$y(s) = \frac{-2}{s+2} + \frac{3}{s+1},$$

$$y(t) = L^{-1}\{y(s)\} = 3L^{-1}\{\frac{1}{s+1}\} - 2L^{-1}\{\frac{1}{s+2}\},$$

$$\therefore \quad y(t) = 3e^{-t} - 2e^{-2t}$$

Example2

Find the solution of the following (d.e) by (L. T)

$y'' + 4y' + 4y = 2$ .... (i)

Which satisfies initial condition?

$y(0) = 1$ and $y'(0) = 1$.

Solution

$L\{y''(t)\} = s^2\{y(s)\} - s\,y(0) - y'(0)$
$L\{y'(t)\} = s\{y(s)\} - y(0).$
$L\{y(t)\} = y(s)$
Since $L\{y\} = y(s)$

} Put in (i)

$$s^2\{y(s)\} - 1 + 4[s\,y(s)] + 4y(s) = \frac{2}{s}$$

$$[s^2 + 4s + 4]y(s) = \frac{2}{s} + 1 = \frac{2+s}{s},$$

$$y(s) = \frac{s+2}{s(s^2+4s+4)} = \frac{s+2}{s(s+2)^2} = \frac{s+2}{s(s+2)},$$

$$y(s) = \frac{s+2}{s(s+2)} = \frac{A}{s} + \frac{B}{s+2},$$

$1 = A(s+2) + Bs,$

$\longrightarrow B = -\frac{1}{2}$ and $A = \frac{1}{2}$,

$$y(s) = \frac{\frac{1}{2}}{s} + \frac{-\frac{1}{2}}{s+2},$$

$y(t) = L^{-1}\{ y(s)\} = 1/2L^{-1}\{\frac{1}{s}\} - 1/2L^{-1}\{\frac{1}{s+2}\}$,

$\therefore\ y(t) = \frac{1}{2} - \frac{1}{2}e^{-2t}$.

Problems

Find the solution of the following (d.e) by (L. T), which satisfies the given initial conditions:-

(1) $y'' + 4y' + 3y = 0$, at $y(0) = 3$ and $y'(0) = 1$,
(2) $y'' + 4y' + 4y = 2$, at $y(0) = 0$ and $y'(0) = 1$,
(3) $y'' - y = 0$, at $y(0) = 0$ and $y'(0) = 1$,
(4) $y'' - 5y' + 6y = 0$, at $y(0) = 0$ and $y'(0) = 1$,
(5) $y'' - 9y = \sin t$, at $y(0) = 1$ and $y'(0) = 0$,
(6) $y'' - 9y = e^{t}$, at $y(0) = 1$ and $y'(0) = 0$,
(7) $y'' + 4y = \sin t$, at $y(0) = 0$ and $y'(0) = 1$,
(8) $y'' + 4y' + 4y = 4\cos 2t$, at $y(0) = 2$ and $y'(0) = 5$,